Norm attainment under finite-dimensional representations

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Residually Finite-Dimensional C^*-algebras

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$$\bigoplus_{\pi\in\mathcal{F}} \pi : A \to \prod_{\pi\in\mathcal{F}} \mathbb{M}_{n_\pi}.$$

Equivalently, A is RFD if for any $a \in A$,

$$\|a\| = \sup_{\substack{\pi\in\mathrm{Irr}(A)\\ \dim(\pi)<\infty}} \|\pi(a)\|.$$

Key Example

Theorem (Choi, 1980)

Let $\mathbb{F}_n$ be the free group on $n \leq \infty$ generators. The full group C^*-algebra $C^*(\mathbb{F}_n)$ is RFD.

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Theorem (Fritz-Netzer-Thom, 2014)

For $n < \infty$ and $a \in \mathbb{C}\mathbb{F}_n$, there exists a finite-dimensional representation π of $C^*(\mathbb{F}_n)$ such that $\|\pi(a)\| = \|a\|_u$.

On a dense subset of $C^*(\mathbb{F}_n)$

In particular, for every $a \in \mathbb{C}\mathbb{F}_n$,

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Corollary

$C^*(\mathbb{F}_n)$ is RFD.

Finite-dimensional norm-attaining elements

For a C^*-algebra A, let $A_{\mathcal{F}}$ denote its subset of elements that attain their norm under a finite-dimensional representation, i.e.

$$A_{\mathcal{F}} = \left\{ a \in A : \|a\| = \max_{\substack{\pi \in \mathrm{Irr}(A) \\ \dim(\pi) < \infty}} \|\pi(a)\| \right\}.$$

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Proposition

For any C^*-algebra A, if $A_{\mathcal{F}}$ is dense in A, then A is RFD.

What about the converse? Is it true that

$$A \text{ RFD} \implies A = \overline{A_{\mathcal{F}}}?$$

Dense subsets

Theorem (C.-Shulman, 2017)

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Proof.

Fun with functional calculus □

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(This is weaker than subhomogeneity.)

Implications for $C^*(\mathbb{F}_n)$

Theorem (Fritz-Netzer-Thom)

For $n < \infty$,

$$\mathbb{C}\mathbb{F}_n \subset C^*(\mathbb{F}_n)_{\mathcal{F}} \subset C^*(\mathbb{F}_n).$$

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Corollary (C.-Shulman, 2017)

For $1 < n < \infty$,

$$\mathbb{C}\mathbb{F}_n \subset C^*(\mathbb{F}_n)_{\mathcal{F}} \subsetneq C^*(\mathbb{F}_n).$$

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1. $A = A_{\mathcal{F}}$.
2. *All irreducible representations of A are finite-dimensional.*
3. *A has no simple infinite-dimensional AF subquotient.*

AF C^*-algebras

Definition

We say a C^*-algebra B is **Approximately Finite-Dimensional** (AF) if it contains a nested sequence of finite-dimensional C^*-subalgebras

$$B_1 \subseteq B_2 \subseteq \dots \subseteq B$$

such that $B = \overline{\bigcup_n B_n}$.

Two key examples:

- The CAR algebra

$$\mathbb{C} \subset \mathbb{M}_2(\mathbb{C}) \subset \mathbb{M}_4(\mathbb{C}) \subset \ldots \subset \mathbb{M}_{2^\infty}$$

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- Compact operators on ℓ^2

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Both of these are *simple* AF C^*-algebras.

Proof Outline

$$A = A_{\mathcal{F}}$$

All irreducible representations of A are finite-dimensional.

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AF Mapping Telescopes

Definition (Brown)

For an AF C^*-algebra B with inductive sequence of finite-dimensional subalgebras (B_n), we define **mapping telescope** $T(B_1, B_2, \ldots)$ (or just $T(B)$) by

$$T(B) = \{f \in C_0((0, \infty], B) |\ f(t) \in B_n\ \forall\ t \leq n\}.$$

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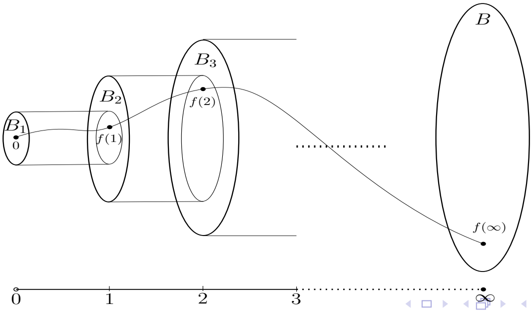

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Lemma

If B is a simple, infinite-dimensional AF algebra, then there exists $f \in T(B) \setminus T(B)_{\mathcal{F}}$ with $\|f\| = \|f(\infty)\|$.

Nice C^*-algebras

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If B is an AF algebra, then $T(B)$ is projective.

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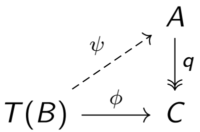
Theorem (Loring-Pedersen, 1998)

If B is an AF algebra, then $T(B)$ is projective.

In other words, for any C^*-algebras A and C with a surjective $*$-homomorphism $q : A \twoheadrightarrow C$, any $*$-homomorphism $\phi : T(B) \to C$ lifts to a $*$-homomorphism $\psi : T(B) \to A$ so that $\phi = q \circ \psi$, giving us a commutative diagram:



Back to the proof

Theorem (C.-Shulman, 2017)

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Simple inf-dim AF subquotient $\Rightarrow A_{\mathcal{F}} \subsetneq A$

Suppose $A_0 \subseteq A$, B simple, infinite-dimensional AF, and $q : A_0 \twoheadrightarrow B$ a surjective $*$-homomorphism.

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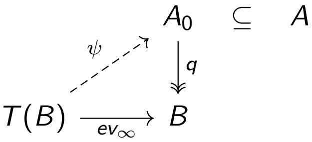

Let $ev_\infty : T(B) \to B$ be the map that sends $f \mapsto f(\infty)$, and let $\psi : T(B) \to A_0$ of ev_∞ be the $*$-homomorphism guaranteed by the projectivity of $T(B)$.

$A = A_{\mathcal{F}} \Rightarrow$ No simple inf-dim AF subquotients

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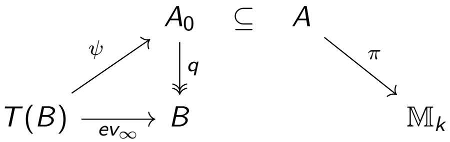

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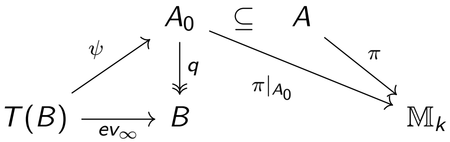

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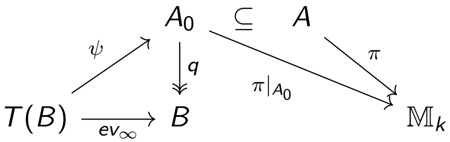

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Suppose $\psi(f) \in A_{\mathcal{F}}$. Then $f \in T(B)_{\mathcal{F}}$. $\square$

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For an RFD C^*-algebra A, the following are equivalent.

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Corollary

For $1 < n < \infty$,

$$\mathbb{C}\mathbb{F}_n \subsetneq C^*(\mathbb{F}_n)_{\mathcal{F}} \subsetneq C^*(\mathbb{F}_n).$$

Sequences of finite-dimensional norms

Assume that A has irreducible representations of all finite dimensions.

Definition

For each $n \in \mathbb{N}$, define the seminorm $\|\cdot\|_{\mathbb{M}_n}$ on A by

$$\|a\|_{\mathbb{M}_n} = \sup\{\|\pi(a)\| : \pi : A \to \mathbb{M}_k, k \leq n\}.$$

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$$\|a\|_{\mathbb{M}_n} = \sup\{\|\pi(a)\| : \pi : A \to \mathbb{M}_k, k \leq n\}.$$

Then for each $a \in A$, we can form the non-decreasing sequence

$$(\|a\|_{\mathbb{M}_n})_{n \in \mathbb{N}} \in \ell^\infty_+.$$

How do your norms grow?

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For example, knowing a C^*-algebra is RFD tells us *that* $\|a\|_{\mathbb{M}_n} \nearrow \|a\|$ as $n \to \infty$ for every $a \in A$.

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Question

Given a non-decreasing sequence $(\lambda_n) \in \ell^\infty_+$, can we find an $a \in A$ with $\|a\|_{\mathbb{M}_n} = \lambda_n$ for each n?

A partial answer

Theorem (C.-Shulman, 2017)

Suppose A has irreducible representations of all finite dimensions. Then for any bounded non-decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of non-negative real numbers that is eventually constant, there exists an $a \in A$ so that

$$(\|a\|_{\mathbb{M}_n})_{n\in\mathbb{N}} = (\lambda_n)_{n\in\mathbb{N}}.$$

Epilogue

Theorem (Shulman, 2017)

The following are equivalent for a C^*-algebra A.

1. *A is type I.*
2. *The spectral radius is continuous on A.*
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(1) $\Rightarrow$ (2) (V. Shulman-Turovskii)
(2) $\Rightarrow$ (3) Contrapositive is clear.
(3) $\Rightarrow$ (1) Lifting argument with $T(\mathbb{M}_{2^\infty})$. $\square$

Thank you.

Some Recommended Reading

K. Courtney and T. Shulman.
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