Amalgamated free products of strongly RFD C^*-algebras over central subalgebras

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Residually finite-fimensional C^*-algebras

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In other words, the direct sum of these representations yields an isometric embedding

$$\bigoplus_{\pi \in \mathcal{F}} \pi : A \to \prod_{\pi \in \mathcal{F}} \mathbb{M}_{n_\pi}.$$

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Theorem (Mal'cev, 1940)

Let G be a discrete group. If G is RF, then it is MAP, and the converse holds when G is finitely generated.

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What about $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$?

More examples?

Permanence Properties: Free Products

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When is the free product of two RF/ MAP groups RF/ MAP?

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Full free products of C^*-algebras

Definition

Given C^*-algebras A_1 and A_2, their **full free product**, $A_1 * A_2$ is the completion of the free $*$-algebra generated by $A_1 \sqcup A_2$ with respect to the largest C^*-norm whose restriction to each A_i yields the original norm.

Full free products of C^*-algebras

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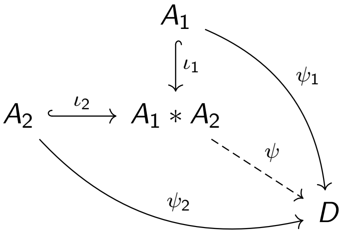

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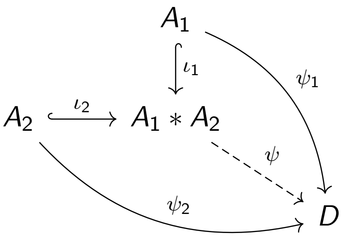

If we assume the A_i, D, and all the maps are unital, then we have the **unital full free product** $A_1 *_{\mathbb{C}} A_2$.

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Corollary (Choi, 1980)

$C^*(\mathbb{F}_2) = C^*(\mathbb{Z} * \mathbb{Z}) = C^*(\mathbb{Z}) *_{\mathbb{C}} C^*(\mathbb{Z})$ *is RFD.*

More examples?

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Amalgamated free products over common subalgebras

For C^*-algebras A_1, A_2, C with embeddings $C \hookrightarrow A_i$, the **amalgamated free product** is a C^*-algebra $A_1 *_C A_2$ together with $*$-homomorphisms $\iota_i : A_i \to A_1 *_C A_2$ such that

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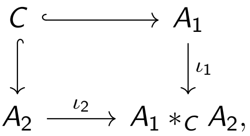

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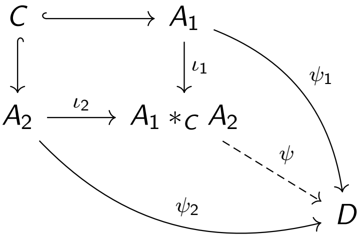

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Some remarks

1. We call $A_1 *_C A_2$ the **pushout** of the following diagram.

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2. (Blackadar '78) The maps $\iota_i : A_i \to A_1 *_C A_2$ are injective.
3. If G_1 and G_2 are discrete groups with common subgroup H, then $C^*(G_1 *_H G_2) \simeq C^*(G_1) *_{C^*(H)} C^*(G_2)$.

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Inspirations and warnings from groups

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(Baumslag, '63) Assume G_1 and G_2 are discrete groups.

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Example (Brown-Dykema, 2004)

Consider the unital embeddings $\mathbb{C} \oplus \mathbb{C} \to \mathbb{M}_2$ and $\mathbb{C} \oplus \mathbb{C} \to \mathbb{M}_3$ given by

$$1 \oplus 0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad 1 \oplus 0 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Then $\mathbb{M}_2 *_{\mathbb{C} \oplus \mathbb{C}} \mathbb{M}_3$ is not finite, which means it cannot be RFD.

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- When A_i are both finite-dimensional, $A_1 *_C A_2$ is RFD iff the A_i have faithful tracial states whose restrictions to C agree. (Brown-Dykema, '04, Armstrong-Dykema-Exel-Li, '04)

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  But if H is central then $G_1 *_H G_2$ is RF.

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Let $C \subseteq A_1, A_2$ be unital inclusions of separable C^*-algebras.

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- If A_i are both commutative, then $A_1 *_C A_2$ is RFD. (Korchagin, '14)

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Theorem (C.-Shulman, 2018)

Let A_1 and A_2 be unital separable RFD C^*-algebras and $C \subset A_1, A_2$ a central subalgebra. Then the amalgamated free product $A_1 *_C A_2$ is RFD when A_1 and A_2 are strongly RFD.

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Definition

A C^*-algebra A is **strongly RFD** if every quotient of A is RFD.

Remarks on the proof

In spirit, we show that each irreducible representation $(\rho, \mathcal{H})$ of $A_1 *_C A_2$ is a pointwise $*$-strong limit of finite-dimensional representations $\sigma_n : A_1 *_C A_2 \to P_n B(\mathcal{H}) P_n$ where $P_n \in B(\mathcal{H})$ are finite-rank projections such that $P_n \nearrow I_{\mathcal{H}}$.

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Remark

If we assume $C = \mathbb{C}$ or $C = 0$, we can drop "strongly" and recover the result of Exel and Loring for separable C^*-algebras.

Examples of strongly RFD C^*-algebras

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4. just-infinite[2] RFD C^*-algebras (Grigorchuk-Musat-Rørdam, '16).

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- (Korchagin, '14) If G_1 and G_2 are abelian groups and H a common subgroup, then $C^*(G_1) *_{C^*(H)} C^*(G_2)$ is RFD.
- (C.-Shulman, '18) If G_1 and G_2 are virtually abelian groups and H is a common central subgroup, then $C^*(G_1) *_{C^*(H)} C^*(G_2)$ is RFD.

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Corollary (C.-Shulman, 2018)

If H is central and $C^*(G_1)$ and $C^*(G_2)$ are separable and strongly RFD, then $G_1 *_H G_2$ is MAP.

Computability

Theorem (Fritz-Netzer-Thom, 2014)

For a finitely presented group G, if $C^*(G)$ is RFD, then the operator norm in the universal unitary representation of G is computable, i.e. there exists an algorithm that allows us to approximate the value

$$\sup\{\|\pi(a)\| : \pi \text{ a unitary representation of } G\},$$

to any precision with rational numbers for any $a \in \mathbb{Z}G$.

Computability

Corollary (Li-Shen, C.-Shulman)

Let G_1, G_2 be finitely presented groups and H a common subgroup. Then the operator norm in the universal unitary representation of $G_1 *_H G_2$ is computable if

1. *H is finite, $G_1 = G_2$, and $C^*(G_1)$ is RFD or*
2. *H is finitely generated and central and $C^*(G_1)$ and $C^*(G_2)$ are strongly RFD.*

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- By Rosenberg's Theorem, they have to be amenable.
- All quotients of finitely generated nilpotent groups are RF, but these have strongly RFD full group C^*-algebras only if they are virtually abelian.

Thanks for your attention!

Some recommended reading

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