The WEP, the LLP, and Some Related Conjectures

Kristin Courtney

1 Introduction

The following are (extended) notes from a four-week lecture series given in the Operator Theory Seminar at the University of Virginia in Fall 2016 on several important properties and results from Eberhard Kirchberg's 1993 paper [9]. Since his seminal work was published, several authors have expanded and simplified some of his arguments and results, notably Gilles Pisier ([16]), Naurataka Ozawa ([14]), and Nate Brown and Naurataka Ozawa([2]). We take advantage of this subsequent material for many of the arguments presented and will often refer the reader to these texts for proofs omitted here.

Unfortunately, four lectures is hardly enough time to fully grasp many of the deep connections among these and other properties and results, so we will direct our focus toward a "goal" theorem and try to restrict and refine the intermediate results accordingly.

Theorem 1. *The following are equivalent:*

1. $C^*(\mathbb{F}_\infty)\otimes_{max}C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty)\otimes_{min}C^*(\mathbb{F}_\infty)$*, where* $\mathbb{F}_\infty$ *is the free group on countably many generators. (Kirchberg's Conjecture)*
2. $C^*(\mathbb{F}_\infty)$ *has the WEP.*
3. *All* C^**-algebras are QWEP. (QWEP Conjecture)*
4. *LLP* $\Rightarrow$ *WEP.*

Time willing, we will mention how these connect to Connes Embedding Conjecture via several "permanence properties" of the WEP and QWEP.

Conjecture 1 (Connes)**.** *Every finite von Neumann algebra with separable predual is embeddable in to an ultrapower* $\mathcal{R}^\omega$ *of the hyperfinite* II_1*-factor* $\mathcal{R}$*.*

Most of our work, however, will be to establish the following tensorial duality between the LLP and WEP:

Theorem 2. *Let* A *and* B *be* C^**-algebras. Then*

1. A *has the LLP* $\Leftrightarrow A \otimes_{\max} B(\ell^2) = A \otimes_{\min} B(\ell^2)$*,*

2. *B has the WEP* $\Leftrightarrow C^*(\mathbb{F}_\infty) \otimes_{\max} B = C^*(\mathbb{F}_\infty) \otimes_{\min} B$*, and*

3. *A has the LLP and B has the WEP* $\Rightarrow A \otimes_{\max} B = A \otimes_{\min} B$.

References will be given to the authors of or inspirations for certain arguments as we go. We will also occasionally restrict arguments to a unital case and provide references (or wave our hands and say that it suffices to prove the claim for the unitization) for the non-unital case. We will avoid restricting ourselves to the separable case unless it is the only case. You will notice a dearth of references to relative weak injectivity (rwi) (also known as weakly cp complementation[1]). This is out of respect for time constraints and quantity of material presented, not disrespect for the subject. In fact, many of the proofs given for characterizations for the WEP (also known as weak injectivity) come from parallel characterizations for relative weak injectivity in [2].

2 Preliminaries

2.1 Tensor Product Norms

Much of this section comes from chapter 3 from [2] and so arguments will not be given that can be found in that text. Deviations will be proved.

Let A, B, and C be C^*-algebras and H a Hilbert space.

We denote an algebraic tensor product (with natural involution) by

$$A \odot B = \left\{ \sum_{i=1}^{n} a_i \otimes b_i : a_i \in A, b_i \in B, \text{ for } 1 \leq i \leq n, n \in \mathbb{N} \right\}.$$

To make this a C^*-algebra, we would like to take its completion with respect to a C^*-norm $\|\cdot\|_\alpha$ (which will actually be cross norms[2]). We denote the C^*-algebra $\overline{A \odot B}^{\|\cdot\|_\alpha} = A \otimes_\alpha B$. However, on this algebra, we can define multiple C^*-norms $\|\cdot\|_\alpha$, which are often distinct. Indeed, we even have a name for C^*-algebras whose algebraic tensor product with any other algebra has a unique C^*-norm.

Definition 1. *A C^*-algebra A is* ***nuclear*** *if $A \odot B$ has a unique C^*-norm for any C^*-algebra B.*

Example 1. *A finite-dimensional C^*-algebra is an easy and often useful example of a nuclear C^*-algebra.*

Though there may exist many distinct C^*-norms on $A \odot B$, two will be of primary interest to us.

Definition 2 (Tensor Product Norms)**.** *For $x = \sum_{i=1}^{n} a_i \otimes b_i \in A \odot B$ (the algebraic tensor product)*

[1] This is a word.

[2] $\|\cdot\|_\alpha$ is a cross norm if $\|a \otimes b\|_\alpha \leq \|a\|_A \|b\|_b$ for any $a \in A$ and $b \in B$.

- $\|x\|_{max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(H_\pi)$ *is an irrep* $\}$
- $\|x\|_{min} = \| \sum_{i=1}^{n} \pi_A(a_i) \otimes \pi_B(b_i) \|_{B(H_A \otimes H_B)}$
  where (π_A, H_A) *and* (π_B, H_B) *are* ***any*** *faithful reps of* A *and* B*, resp.*

The completion $A \otimes_{\max} B = \overline{A \odot B}^{\|\cdot\|_{\max}}$ *is often called the* ***projective*** ***C^*-tensor product****; the completion* $A \otimes_{\min} B = \overline{A \odot B}^{\|\cdot\|_{\min}}$ *is often called the* ***injective C^*-tensor product****.*

Theorem 3 (Universal Property of $\otimes_{\max}$)**.** *For any C^*-algebra C, any *-homomorphism $A \odot B \to C$ extends uniquely to a *-homomorphism $A \otimes_{max} B \to C$.*
*In particular, if $\pi_A : A \to C$ and $\pi_B : B \to C$ are *-homomorphisms with commuting ranges, then the map $\pi_A \times \pi_B$ extends uniquely to $A \otimes_{max} B$.*

Theorem 4 (Restrictions of Tensor Product Maps)**.** *If $\pi : A \odot B \to B(H)$ is a *-homomorphism, then there exist restrictions*

$$\pi_A : A \to B(H) \text{ and } \pi_B : B \to B(H)$$

with commuting ranges such that $\pi = \pi_A \times \pi_B$
(where $\pi_A \times \pi_B(a \otimes b) = \pi_A(a)\pi_B(b)$).
If A and B are unital, we may replace $B(H)$ with any unital C^*-algebra C.

Thanks to the universality of $\| \cdot \|_{\max}$ and a theorem due to Takesaki (see [17] or [2]), we know these are truly norms and that for any $x \in A \odot B$ and any C^*-norm $\| \cdot \|_\alpha$ on $A \odot B$,

$$\|x\|_{\min} \leq \|x\|_\alpha \leq \|x\|_{\max}.$$

More formally, we have natural surjective *-homomorphisms

$$A \otimes_{\max} B \to A \otimes_{\alpha} B \to A \otimes_{\min} B.$$

Consequently, to prove that $A \odot B$ has a unique C^*-norm, it suffices to show that

$$A \otimes_{\min} B = A \otimes_{\max} B.$$

We now give two important examples, which are both deep theorems in themselves.

Example 2 (Kirchberg, [9])**.** *Let $\mathbb{F}$ be any free group with full group C^*-algebra $C^*(\mathbb{F})$ and H any Hilbert space. Then*

$$C^*(\mathbb{F}) \otimes_{\max} B(H) = C^*(\mathbb{F}) \otimes_{\min} B(H).$$

Example 3 (Junge, Pisier, [7])**.** *If H is an infinite dimensional Hilbert space, then $B(H) \otimes_{\max} B(H) \neq B(H) \otimes_{\min} B(H)$*

To gain a greater appreciation for having a unique tensor norm on $A \odot B$, consider the following intuitive property for $\otimes_{\min}$ that can fail for $\otimes_{\max}$:

Proposition 1. *If $A \subseteq B$ as C^*-algebras and C is another C^*-algebra, then*

$$A \otimes_{\min} C \subseteq B \otimes_{\min} C.$$

Indeed, this means exactly that the norm on $A \odot C$ inherited as a subset of $B \otimes_{\min} C$ is the same as then $\|\cdot\|_{\min}$ norm on $A \odot C$, which follows from the fact that $\|\cdot\|_{\min}$ is independent of faithful representation (and a faithful representation of A can be obtained by restricting a faithful representation of B).

However, this can fail for $\otimes_{\max}$. It is perhaps more informative here to understand *why* there can be counterexamples than to see actual counterexamples. To that end, notice that we automatically have the containment $A \odot C \subseteq B \odot C$. Hence, a representation on the larger algebra restricts to a representation on the smaller algebra, but there may be representations on $A \odot C$ that do not extend to $B \odot C$. This means that, for $x \in A \odot C$,

$$\|x\|_{A \otimes_{\max} C} \geq \|x\|_{B \otimes_{\max} C}.$$

As would be expected, it turns out to be quite significant when this property does hold for $\otimes_{\max}$ for C^*-algebras $A \subseteq B$, and C;[3] it also turns out to be quite significant for this property to hold for a given C^*-algebra A no matter our choice of $B \supseteq A$ and C, so much so that it gets a name.

Definition 3. *A C^*-algebra A has the* ***Weak Expectation Property*** *(WEP) if for any C^*-algebra B containing A and any C^*-algebra C,*

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C.$$

If this definition does not meet your expectations, just wait. Nonetheless, now we can see that nuclearity is at least as strong as the WEP.

Proposition 2. *If A is nuclear, then A has the WEP.*

Proof. Suppose A is nuclear and B and C are C^*-algebras with $A \subseteq B$. Then

$$A \otimes_{\max} C = A \otimes_{\min} C \subseteq B \otimes_{\min} C.$$

That is, for all $x \in A \odot C$,

$$\|x\|_{B \otimes_{\max} C} \leq \|x\|_{A \otimes_{\max} C} = \|x\|_{B \otimes_{\min} C} \leq \|x\|_{B \otimes_{\max} C}.$$

□

Before we depart, however, another couple of extremely useful and rather straightforward facts for $\otimes_{\max}$ and $\otimes_{\min}$ must be stated. It is just as unnatural for them to be inserted here as any where else in the narrative, and having them here will prevent clutter when we wish to use them (repeatedly) later.

[3] This is that relative weak injectivity we said would be underrepresented.

Theorem 5 (Continuity of Tensor Product Maps)**.** [4]

If $\phi : A \to C$ and $\psi : B \to D$ are cp[5] *maps, then $\phi \odot \psi : A \odot B \to C \odot D$ extends to*

- *a cp map $\phi \otimes_{max} \psi : A \otimes_{max} B \to C \otimes_{max} D$ and*
- *a cp map $\phi \otimes_{min} \psi : A \otimes_{min} B \to C \otimes_{min} D$.*

Proposition 3. *For any C^*-algebras A and B,*

$$A \otimes_{\max} B \simeq B \otimes_{\max} A$$
$$A \otimes_{\min} B \simeq B \otimes_{\min} A$$

2.2 Conditional Expectations

Before we are ready to define a weak conditional expectation, it would help to recall the definition of a conditional expectation.

Definition 4. *Given C^*-algebras $A \subseteq B$, a* ***conditional expectation*** *from B to A is a linear projection $\phi : B \to A$ (i.e. $\phi|_A = id_A$) such that ϕ is an A-bimodule map (i.e. $\phi(aba') = a\phi(b)a'$ for any $a, a' \in A$ and $b \in B$).*[6]

It will be more helpful to recall Tomiyama's characterization of conditional expectations.

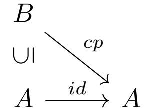
Theorem 6 (Conditional Expectation–Tomiyama)**.** *Given C^*-algebras $A \subseteq B$ and a linear projection $\phi : B \to A$ the following are equivalent*

1. *ϕ is a contitional expectation*
2. *ϕ is contractive and completely positive (ccp)*
3. *ϕ is contractive*

An important example comes from injective C^*-algebras.

Definition 5. *A (unital) C^*-algebra A is* ***injective*** *(in the category of (unital) operator systems with (u)cp maps) if for every C^*-algebra C and operator system $S \subseteq C$, any (u)cp map $\phi : S \to A$ can be extended to a (u)cp map $\tilde{\phi} : C \to A$.*

In particular, if A and B are C^*-algebras with A injective, then we have a conditional expectation from B onto A:



[4] This result is sometimes referred to as the functoriality of $\otimes_{\max}$ and $\otimes_{\min}$.

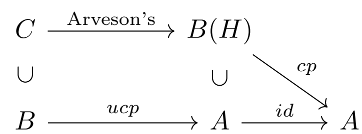
[5] Complete positivity is crucial, lest you end up with an unbounded map.

[6] Yes, this is the non-commutative version of a conditional expectation, and yes, it is inspired by free probability.

The following will be used relentlessly and will be referred to as either "Arveson's Theorem" or "the injectivity of $B(H)$":

Theorem 7 (Arveson's Extension Theorem)**.** *For any Hilbert space H, $B(H)$ is injective in the category of (unital) operator systems with (u)cp maps.*

We can use Arveson's Extension Theorem to show that for any C^*-algebras $A \subseteq B$ where A is faithfully embedded in $B(H)$ for some Hilbert space H, the injectivity of A is equivalent to the existence of a conditional expectation from $B(H)$ onto A. (Indeed, if A is injective, then the identity on A extends to a conditional expectation. On the other hand, if there is such a conditional expectation, then we follow the following diagram.)



Now, suppose that, given $A \subseteq B$, there is, not a cp projection from B onto A but a cp map from B onto the enveloping von Neumann algebra of A (i.e. weak closure of the image of A under the universal representation $(\pi_u, B(H_u))$) that agrees with π_u on A. Since B is mapping to the weak closure of the universal representation of A as opposed to A, we call the map a weak conditional expectation. For convenience, we identify $A^{**} = \pi_u(A)''$ and often treat A as a subalgebra of A^{**} and A^{**} as a subalgebra of $B(H_u)$.

Definition 6. *For C^*-algebras $A \subseteq B$, a **weak conditional expectation** is a cp map $\phi : B \to A^{**} \simeq \pi_u(A)''$ such that ϕ agrees with the natural embedding of A into A^{**}.*

Now, we are ready to give, what we will henceforth take to be, the definition of the Weak Expectation Property:

Definition 7 (WEP)**.** *Let A be a C^*-algebra, and assume $A \subseteq A^{**} \subseteq B(H_u)$ where $\pi_u : A \to B(H_u)$ is the universal representation of A given by the Gelfand-Naimark theorem. A has the **Weak Expectation Property** (WEP) if there exists a ccp[7] map $\phi : B(H_u) \to A^{**}$ such that $\phi(a) = a$ for all $a \in A$.*

Remark 1. *For the reader familiar with approximate injectivity, it is a highly non-trivial fact that WEP is not the same as approximate injectivity from [6]. Indeed, Kirchberg ([9]) proved that these two are the same iff $B(\ell^2) \odot B(\ell^2)$ has a unique tensor norm, the latter statement being disproven by Junge and Pisier in [7].*

Definition 8 (QWEP)**.** *A C^*-algebra is **QWEP** if it is the quotient of a C^*-algebra with the WEP.*

[7] In [2], this map is ucp without assumption that A is unital. In [16], it is not ucp. We are unable to establish the soon to follow equivalent characterizations if the definition requires a ucp map even when A is non-unital.

3 WEP

The material in this section will be a veritable Frankenstein's monster built of several sources, folklore, and observations. Readily sourced sources will be cited; folklore will not; deviations from sources will be proved.

The WEP has been characterized dozens of times in various contexts. We will show restraint and only use the characterizations we need. First, we should wean ourselves of the universal representation. To do this, we will need Arveson's extension theorem and the universal property of the double dual.

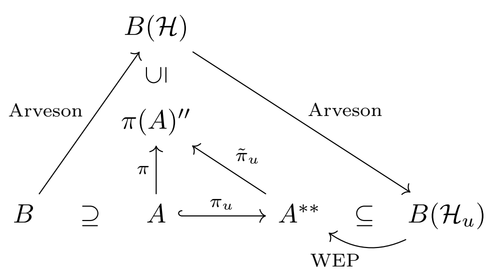
Proposition 4. *For each non-degenerate representation $\pi : A \to B(H)$ of A, there exists a unique normal extension $\tilde{\pi} : A^{**} \to B(H)$ such that $\pi(A^{**}) = \pi(A)''$.*

Proposition 5. *Let A be a C^*-algebra. TFAE:*

1. *A has the WEP (i.e. there is a weak conditional expectation from $B(H_u)$ onto A^{**} that agrees with id_A).*
2. *For any C^*-algebra B such that A embeds into B, there exists a ccp map $\psi : B \to A^{**}$ such that $\psi(a) = a$ for all $a \in A$.*
3. *For any C^*-algebra B such that A embeds into B and any $*$-homomorphism $\pi : A \to B(H)$ there exists a ccp map $\rho : B \to \pi(A)''$ such that $\rho(a) = \pi(a)$ for all $a \in A$.*

Remark 2. *This is an adaptation of Prop 3.6.6 in [2].*

Proof. The proof for (1)⇒(2)⇒(3) involves the universal property of A^{**} and a couple applications of Arveson's Extension Theorem. It can be reduced to the following commutative diagram where $\tilde{\pi}$ is the normal extension of π, and $\pi_u : A \to B(\mathcal{H}_u)$ is the universal representation, where we assume $A^{**} \simeq \pi_u(A)''$ is embedded in $B(\mathcal{H}_u)$.



□

Example 4. *By injectivity, $B(H)$ has the WEP for any Hilbert space H. (In fact, if $B(H) \subseteq B$, the identity on $B(H)$ will extend to a conditional expectation $B \to B(H)$.) However, if H is infinite dimensional, $B(H)$ is non-nuclear. Hence, nuclearity is strictly stronger than the having the WEP.*

Example 5 ([16])**.** *Moreover, a von Neuamnn algebra M has the WEP iff it is injective (iff it is hyperfinite [3]).*

Proof. Necessity follows just as it did for $B(H)$. To see why $M \subseteq B(H)$ having the WEP is sufficient for injectivity, assume S is an operator system inside a C^*-algebra A and $\phi : S \to M$. Then, we have the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\text{Arveson's}} & B(H) & & & & \\ \cup & & \cup & \searrow^{WEP} & & & \\ S & \xrightarrow{\phi} & M & \hookrightarrow & M^{**} & \twoheadrightarrow & M'' = M \end{array}$$

□

This leads us to another important example:

Example 6. *The hyperfinite II_1-factor R has the WEP. Moreover, Kirchberg ([9]) showed that the direct product of C^*-algebras with the WEP also has the WEP; hence, $\prod_{\mathbb{N}} R = \ell^\infty(R)$ has the WEP. Therefore, R^ω is QWEP for any (non-trivial) ultrafilter ω of $\mathbb{N}$.*

Remark 3. *This is why the Weak Expectation Property is sometimes called "Weak Injectivity". (Recall also that A^{**} is injective whenever A is nuclear. See Theorem 11.6 in [16].)*

Recall that a trace τ on $A \subset B(\mathcal{H})$ is *amenable* if there exists a cpc map $\phi : B(\mathcal{H}) \to \pi_\tau(A)''$ such that $\phi(a) = \pi_\tau(a)$ for all $a \in A$.

Corollary 1. *If a unital C^*-algebra A has WEP, then any trace on A is amenable.*

Now, let us see from whence that tensor product characterization (due to Lance, [12]) came:

Theorem 8. *Let A be a C^*-algebra. Then TFAE:*

1. *A has the WEP.*
2. *For any C^*-algebra B with $A \subseteq B$ and any C^*-algebra C,*

$$A \otimes_{max} C \subseteq B \otimes_{max} C$$

3. *For any embedding $A \subseteq B(H)$*

$$A \otimes_{max} C^*(\mathbb{F}_\infty) \subseteq B(H) \otimes_{max} C^*(\mathbb{F}_\infty)$$

4. *For any embedding $A \subseteq B(H)$ and any free group $\mathbb{F}$*

$$A \otimes_{max} C^*(\mathbb{F}) \subseteq B(H) \otimes_{max} C^*(\mathbb{F})$$

Before we begin the proof, we give a few results that will serve as Lemmas. For the (3)⇒(4) portion, we will use Proposition 8.8 from [16].

Lemma 1 (Prop 8.8). *[16] For groups $\Gamma \subseteq G$, there exists a canonical embedding $C^*(\Gamma) \hookrightarrow C^*(G)$ and a conditional expectation from $C^*(G)$ onto the image of $C^*(\Gamma)$ under this embedding.*

It will be prudent to go ahead and flesh out the relevant corollary (which was given in [16] but not explicitly as a corollary).

Corollary 2. *Let $\mathbb{F}$ be an arbitrary free group, then for any finite dimensional subspace $D \subseteq C^*(\mathbb{F})$, there is a C^*-subalgebra C of $C^*(\mathbb{F})$ containing D, which is *-isomorphic to $C^*(\mathbb{F}_\infty)$, and a conditional expectation $C^*(\mathbb{F}) \to C$. In other words, $id_{C^*(\mathbb{F}_\infty)}$ factors through $C^*(\mathbb{F})$ via ucp maps.*

Indeed, since any element in $C^*(\mathbb{F})$ can be expressed with countably many elements $\{g_i\}_{i=1}^\infty$ in $\mathbb{F}$, we have countably many generators to which we can map the generators of $C^*(\mathbb{F}_\infty)$. By norm considerations, this is an embedding. Furthermore, we have a conditional expectation from $C^*(\mathbb{F})$ to $C^*(\langle \{g_i\}_{i=1}^\infty \rangle) \simeq C^*(\mathbb{F}_\infty)$.

For the (4)⇒(1) portion, we will invoke The Trick from [2], so we shall give its proof first. The Trick is originally proved for any C^*-norm on a tensor product that satisfies a couple of properties that are satisfied by $\|\cdot\|_{\max}$. Since we will only need it for this norm, we will state the proposition in terms of this norm.

Lemma 2 (The Trick). *[2] Let $A \subseteq B$ and C be C^*-algebras such that*

$$A \otimes_{max} C \subseteq B \otimes_{max} C.$$

Given representations $\pi_A : A \to B(H)$ and $\pi_C : C \to B(H)$ with commuting ranges, there exists a ccp map $\phi : B \to \pi_C(C)'$ which extends π_A.

We give the proof of The Trick in the unital case. The non-unital case follows from the unital case after some observations on induced inclusions in the tensor product setting. See [2] for the rest of the proof.

Proof. [2] Assume that A, B, and C are unital with $1_A = 1_B$. By universality, $\pi_A \times \pi_C : A \odot C \to B(H)$ extends to a *-homomorphism $\pi_A \times \pi_C : A \otimes_{max} C \to B(H)$. Since

$$A \otimes_{max} C \subseteq B \otimes_{max} C,$$

$\pi_A \times \pi_C$ extends to a ucp map $\Phi : B \otimes_{max} C \to B(H)$ by injectivity of $B(H)$. Let $\phi : B \to B(H)$ be given by $\phi(b) = \Phi(b \otimes 1_C)$.

To see that $\phi(B) \subseteq \pi_C(C)'$, notice that $\Phi_{\mathbb{C}1_B \otimes C} = \pi_C$ is a *-homomorphism, and so $\mathbb{C}1_B \otimes C$[8] is in the multiplicative domain[9] of Φ. Furthermore, $B \otimes \mathbb{C}1_C \subseteq$

[8]Recall that finite dimensional C^*-algebras are nuclear, so there is only one tensor product norm

[9]The multiplicative domain of Φ is $\{x \in B \otimes_{max} C : \Phi(xy) = \Phi(x)\Phi(y)\ \&\ \Phi(yx) = \Phi(y)\Phi(x)\ \forall\ y \in B \otimes_{max} C\}$, or the largest subspace X such that $\Phi|_X$ is a $*$-homomorphism.

$(\mathbb{C}1_B \otimes C)$. So, for $b \in B$ and $c \in C$,

$$\begin{aligned}\phi(b)\pi(c) &= \Phi(b \otimes 1_C)\Phi(1_B \otimes C) \\ &= \Phi((b \otimes 1_C)(1_B \otimes C)) \\ &= \Phi((1_B \otimes c)(b \otimes 1_C)) \\ &= \pi(c)\phi(b).\end{aligned}$$

□

Now, we are ready to give the proof of the theorem.

Proof of Theorem 8. For (1)⇒(2), we follow the proof for Prop 3.6.2 in [2]:

(1)⇒(2) Suppose A, B, and C are C^*-algebras where A has the WEP and $A \subseteq B$. First note that, by the universal property of $\otimes_{\max}$, the embedding $A \odot B \hookrightarrow B \otimes_{\max} C$ extends to a *-homomorphism

$$\rho : A \otimes_{\max} B \to B \otimes_{\max} C.$$

We have only to show that this map is injective. To do so, we will take a faithful representation $\pi : A \otimes_{\max} C \to B(H)$ and show that ρ is a factor of π. Let $\pi_A : A \to B(H)$ and $\pi_C : C \to B(H)$ be the restrictions of π with commuting ranges. In particular, $\pi_C(C)$ and $\pi_A(A)''$ are two commuting subspaces of $B(H)$ and hence their natural inclusion maps $\iota_{\pi_C(C)}$ and $\iota_{\pi_A(A)''}$ are two *-homomorphisms with commuting ranges. Again, the universal property of $\otimes_{\max}$ allows us to extend $\iota_{\pi_C(C)} \times \iota_{\pi_A(A)''}$ to a *-homomorphism

$$\pi_A(A)'' \otimes_{\max} \pi_C(C) \to B(H).$$

Since A has the WEP and $A \subseteq B$, there is a ccp extension $\phi : B \to \pi_A(A)''$ of π_A. Then, by functoriality of $\otimes_{\max}$, $\phi \odot \pi_C$ extends to a ccp map

$$\phi \otimes \pi_C : B \otimes_{\max} C \to \pi_A(A)'' \otimes_{\max} \pi_C(C).$$

Then, $\pi = (\iota_{\pi_C(C)} \times \iota_{\pi_A(A)''}) \circ (\phi \otimes \pi) \circ \rho$, and thus ρ is injective.

(3)⇒(4) Let $\mathbb{F}$ be any free group, $A \subseteq B(H)$ an embedding, and $t = \sum_{i=1}^n a_i \otimes x_i \in A \odot C^*(\mathbb{F})$. Then, by Lemma (2), there is a copy of $C^*(\mathbb{F}_\infty)$ in $C^*(\mathbb{F})$ containing $\{x_1, ..., x_n\}$ (which we will identify with $C^*(\mathbb{F}_\infty)$ for simplicity of notation), and there is a conditional expectation from $C^*(\mathbb{F})$ onto this copy of $C^*(\mathbb{F}_\infty)$. Notice that our proof of (1)⇒(2) shows that, for any C^*-algebra C,

$$C \otimes_{\max} C^*(\mathbb{F}_\infty) \subseteq C \otimes_{\max} C^*(\mathbb{F}).$$

Interpreting this in terms of norms, we have

$$\|t\|_{A \otimes_{\max} C^*(\mathbb{F})} = \|t\|_{A \otimes_{\max} C^*(\mathbb{F}_\infty)} = \|t\|_{B(H) \otimes_{\max} C^*(\mathbb{F}_\infty)} = \|t\|_{B(H) \otimes_{\max} C^*(\mathbb{F})}.$$

(4)⇒(1) Embed $\pi_u(A) \subseteq A^{**} \subseteq B(H_u)$ where $(\pi_u, B(H_u))$ is the universal representation for A. Let $\mathbb{F}$ be a free group on $|U(\pi_u(A)')|$ generators, and let $\pi : C^*(\mathbb{F}) \to \pi_u(A)' \subseteq B(H_u)$ be the surjective *-homomorphism induced by mapping generators of $\mathbb{F}$ to the unitaries of $\pi_u(A)'$. Now, we use The Trick with $C = C^*(\mathbb{F})$, $B = B(H_u)$, $B(H) = B(H_u)$, $\pi_A = \pi_u$, and $\pi_C = \pi$. Then, we get a ccp map $\phi : B(H_u) \to \pi_u(A)'' = A^{**}$.

□

Remark 4. *Pisier's Proposition 8.8 (or, more specifically Lemma 2) is often what is cited in the text when authors have a preferred free group for a particular problem. For instance, knowing that the identity for $C^*(\mathbb{F}_\infty)$ factors through $C^*(\mathbb{F}_2)$ via ucp maps allows one to restate Kirchberg's conjecture and the equivalent conjectures with $C^*(\mathbb{F}_2)$ instead of $C^*(\mathbb{F}_\infty)$.*

We are now well poised to establish the WEP portion of the tensorial duality between the WEP and LLP, i.e.

Theorem 9. *For any C^*-algebra A,*

$$A \text{ has the WEP } \Leftrightarrow C^*(\mathbb{F}_\infty) \otimes_{\max} A = C^*(\mathbb{F}_\infty) \otimes_{\min} A.$$

Before we give the proof, we recall our earlier example 2 where we gave Kirchberg's deep and crucial result:

Theorem 10 (Kirchberg)**.** *For any free group $\mathbb{F}$ and any Hilbert space H,*

$$C^*(\mathbb{F}) \otimes_{\max} B(H) = C^*(\mathbb{F}) \otimes_{\min} B(H).$$

We leave the proof for the appendix.

Proof of Theorem 9. Let $A \subseteq B(H)$ be a faithful embedding. Then, by Kirchberg's Theorem and the fact that $A \otimes_{\min} C^*(\mathbb{F}_\infty) \subseteq B(H) \otimes_{\min} C^*(\mathbb{F}_\infty)$, we can see the equivalence of the two statements with the following diagram:

$$\begin{array}{ccc} B(H) \otimes_{\max} C^*(\mathbb{F}_\infty) & = & B(H) \otimes_{\min} C^*(\mathbb{F}_\infty) \\ \cup\ (\Rightarrow) & & \cup \\ A \otimes_{\max} C^*(\mathbb{F}_\infty) & \overset{(\Leftarrow)}{=} & A \otimes_{\min} C^*(\mathbb{F}_\infty) \end{array}$$

□

4 LLP

Definition 9. *Let A and B be C^*-algebras and J a closed two-sided ideal in B with quotient map $\pi : B \to B/J$. A ccp map $\phi : A \to B/J$ is **liftable** if there is a ccp map $\psi : A \to B$ such that $\pi \circ \psi = \phi$. If A is unital, we say a ccp map $\phi : A \to B/J$ is **locally liftable** if for any finite dimensional operator system $S \subseteq A$ there is a ccp map $\psi : S \to B$ such that $\pi \circ \psi = \phi|_S$.*

Definition 10 (LLP)**.** *A unital C^* algebra A has the* ***(local) lifting property*** *(L)LP if any ucp*[10] *map from A into a quotient C^*-algebra is (locally) liftable. (A non-unital C^*-algebra has (L)LP iff its unitization does.)*

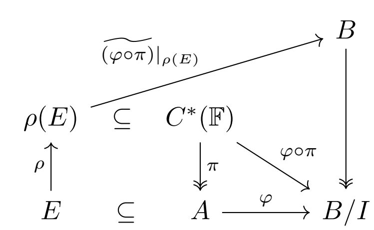
Remark 5. *Drawing a diagram should justify that the LLP is preserved by $*$-isomorphisms. We will frequently replace a C^*-algebra with an isomorphic C^*-algebra for the sake of simplicity.*

The Choi-Effros lifting theorem tells us that all nuclear C*-algebras have the LP. The next important family of examples comes from Kirchberg. A simplified version of the proof can be found in [14].

Theorem 11. *[10] For any free group $\mathbb{F}$, $C^*(\mathbb{F})$ has the LLP. If $\mathbb{F}$ is countably generated, then $C^*(\mathbb{F})$ has the LP.*

Proposition 6. *Let A be a C^*-algebra and $\mathbb{F}$ a free group such that A can be identified with a quotient $C^*(\mathbb{F})/J$ of $C^*(\mathbb{F})$. Then A has the LLP iff the identity on $C^*(\mathbb{F})/J$ is locally liftable.*

Proof. For simplicity, identify $A = C^*(\mathbb{F})/J$, and let $\pi : C^*(\mathbb{F}) \to C^*(\mathbb{F})/J$ be the quotient map. Let $E \subseteq A$ be a finite dimensional operator system and $\rho : E \to C^*(\mathbb{F})$ the lift of $id_A|_E$ guaranteed by assumption. Let B a C^*-algebra with closed two-sided ideal I, and $\varphi : A \to B/I$ a ucp map. Then, $\varphi \circ \pi : C^*(\mathbb{F}) \to B/I$ is a ucp map and $\rho(E) \subseteq C^*(\mathbb{F})$ is a finite dimensional operator system. Then, since $C^*(\mathbb{F})$ has the LLP, there is a lift $\widetilde{(\varphi \circ \pi)|_{\rho(E)}}$ of $(\varphi \circ \pi)|_{\rho(E)}$ to B.



□

Remark 6. *In fact, Kirchberg showed that for $\mathbb{F}$ countable, $C^*(\mathbb{F})$ has the Lifting Property (i.e. any ucp map $C^*(\mathbb{F}) \to B/J$ lifts to a ucp map $C^*(\mathbb{F}) \to B$). So, we can argue similarly that any separable C^*-algebra A has the (Local) Lifting Property iff its identification with a quotient of $C^*(\mathbb{F}_\infty)$ is (locally) liftable. Unfortunately, the proof that $C^*(\mathbb{F}_\infty)$ has the LP is a bit much for these talks, so see [2] for Kirchberg's argument.*

[10] Yes, we did mysteriously shift from ccp maps to ucp maps, but it turns out to be sufficient to restrict ourselves to ucp maps. See [2] 13.1.2 for an argument.

The upshot of this corollary is that, to show that any ucp map lifts, we need only show that one ucp map (in fact *-homomorphism) lifts. With this, we can give a class of examples.

Definition 11. *[13] A C^*-algebra A is projective (with respect to *-homomorphisms) if given any C^*-algebra B with two-sided ideal J, any *-homomorphism $\psi : A \to B/J$ lifts to a *-homorphism $\tilde{\psi} : A \to B$.*

Example 7. *Projective C^*-algebras have the LLP. (Separable projective C^*-algebras have the LP.)*

The proof uses the corollary from above and the observation that the identity is a *-homomorphism.

Remark 7. *Recall that the WEP can be characterized as a weak injectivity property, and, furthermore, any injective C^*-algebra has the WEP. We now see that the LLP is a weakening of projectivity (for *-homomorphisms– it's clearly a weaking of projectivity for ucp maps). If you are curious whether WEP implies LLP or LLP implies WEP, you are asking the right questions.*

After a few theorems, it will be easier to justify some examples and non-examples. However, known examples are much more readily available than known non-examples. For instance, in [15], Ozawa proved the existence of groups whose full C*-algebra does not have the LLP, but no concrete examples were known until Andreas Thom's 2008 example ([18]) of hyperlinear groups that are not residually finite.[11] However, we can at least offer the argument from [1] for why this property should be rare. Indeed, on one hand, every finite dimensional subspace of a C^*-algebra with the LLP can be identified with a subspace of $C^*(\mathbb{F}_\infty)$, and, since $C^*(\mathbb{F}_\infty)$ is separable, the set of n-dimensional operator spaces contained in some C^*-algebra with the LLP is separable. On the other hand, the set of n-dimensional operator spaces is not separable. So, "most" finite dimensional operator spaces will generate a C^*-algebra without the LLP.

The following Effros-Haagerup lifting theorem has proven a crucial characterization of local liftability (and the LLP with some rephrasing and additional propositions).

[11] Because Thom does not spell out the argument for why these group C*-algebras lack the LLP in [18], we sketch it here and outsource relevant definitions. For a group G, let τ_λ denote the trace on $C^*(G)$ coming from its left regular representation. We say a group G is hyperlinear if τ_λ is hyperlinear, meaning there exists a $*$-homomorphism $\rho : C^*(G) \to R^\omega$ such that $\tau_\lambda = \tau_\omega \rho$. We say G has Kirchberg's factorization property (F) if τ_λ is amenable (see [2, Theorem 6.4.3]), meaning there exists a $*$-homomorphism $\rho : C^*(G) \to R^\omega$ such that $\tau_\lambda = \tau_\omega \rho$ and such that ρ has a ucp lift to $\ell^\infty(R)$. The full C*-algebra of a hyperlinear group without (F) would fail to have the LLP. (Indeed, since $\mathbb{R}^\omega$ is QWEP, if $C^*(G)$ had the LLP, then by [14, Corollary 3.12], any ucp map into R^ω would have a ucp lift– not just local lift.) Kirchberg showed in [11] that in the presence of Kazhdan's Property (T), the factorization property (F) and residual finiteness (RF) are the same. Hence Thom's example is hyperlinear without property (F).

Theorem 12 (Effros-Haagerup lifting theorem, '85)**.** *[14] For C^*-algebras A and B with J a closed two-sided ideal of B, a ucp map $\phi : A \to B/J$ is locally liftable iff*

$$\phi \otimes id : A \odot B(\ell^2) \to (B \otimes_{min} B(\ell^2))/(J \otimes_{min} B(\ell^2))$$

is continuous with respect to the minimal tensor norm on $A \odot B(\ell^2)$.

Recall by Theorem 5 that $\phi \odot id$ extends to a ucp map

$$\phi \otimes id : A \otimes_{min} B(\ell^2) \to (B/J) \otimes_{min} B(\ell^2).$$

However, we do not know that

$$(B \otimes_{min} B(\ell^2))/(J \otimes_{min} B(\ell^2)) = (B/J) \otimes_{min} B(\ell^2).$$

In general, we know the following.

Proposition 7. *[2] For C^*-algebras C and B with J a closed two-sided ideal of B, there is a C^*-norm $\|\cdot\|_\alpha$ on $C \odot (B/J)$ such that*

$$(B/J) \otimes_\alpha C \simeq (B \otimes_{min} C)/(J \otimes_{min} C).$$

With this in mind, we can see immediately that, if $(B/J) \otimes B(\ell^2)$ has a unique C^*-norm, then any ucp map $\phi : A \to B/J$ lifts. On the other hand, if $A \odot B(\ell^2)$ has a unique C^*-norm, then any ucp map $\phi : A \to B/J$ lifts as well.

Proposition 8. *Let A be a C^*-algebra. If $A \otimes_{\max} B(\ell^2) = A \otimes_{\min} B(\ell^2)$, then A has the LLP.*

Proof. We assume A is unital or prove the claim for $\tilde{A}$. Let B be a C^*-algebra with closed two-sided ideal J and $\phi : A \to B/J$ a ucp map. By E-H, it (more than) suffices to show that $\phi \odot A \odot B(\ell^2) \to (B/J) \otimes_\alpha B(\ell^2)$ is continuous with respect to $\|\cdot\|_{A \otimes_{\min} B(\ell^2)}$ for any C^*-norm $\|\cdot\|_\alpha$ on $(B/J) \odot B(\ell^2)$. By functoriality and universality of $\otimes_{\max}$, we have a ucp map

$$\phi \otimes_{\max} id_{B(\ell^2)} : A \otimes_{\max} B(\ell^2) \to (B/J) \otimes_{\max} B(\ell^2)$$

extending $\phi \odot id_{B(\ell^2)}$ and a surjective *-homomorphism

$$(B/J) \otimes_{\max} B(\ell^2) \to (B/J) \otimes_\alpha B(\ell^2)$$

extending $id_{(B/J) \odot B(\ell^2)}$. The composition of these maps and the identification

$$A \otimes_{\max} B(\ell^2) = A \otimes_{\min} B(\ell^2)$$

gives us that $\phi \otimes id$ is ucp (and hence, continuous with respect to $\|\cdot\|_{\min}$). □

Here's another perspective on the argument, starting with a different phrasing for the Effros-Haagerup lifting theorem.

Theorem 13 (Effros-Haagerup lifting theorem)**.** *Let B be a* C**-algebra and J a two-sided closed ideal in B with quotient map $\pi : B \to B/J$. The following are equivalent:*

1. *For any* C**-algebra A, the following sequence is exact*
$$0 \to A \otimes_{\min} J \to A \otimes_{\min} B \to A \otimes_{\min} B/J \to 0.$$
2. *The same but for just $A = B(\ell^2)$.*
3. *$id_{B/J}$ is locally liftable.*

Rephrasing this in terms of the LLP, we have the following corollary, which should be compared with exactness for a C*-algebra.

Corollary 3. *If C has the LLP, then for any extension*
$$0 \to J \to E \to C \to 0,$$
and any C**-algebra A, the sequence*
$$0 \to A \otimes_{\min} J \to A \otimes_{\min} E \to A \otimes_{\min} C \to 0$$
is exact.

In particular, consider a C*-algebra A and free group $\mathbb{F}$ such that $C^*(\mathbb{F})$ surjects onto A with kernel $J \subset C^*(\mathbb{F})$. If A has the LLP, then the sequence
$$0 \to B(\ell^2) \otimes_{\min} J \to B(\ell^2) \otimes_{\min} C^*(F) \to B(\ell^2) \otimes_{\min} A \to 0$$
is exact. By Kirchberg's theorem $B(\ell^2) \odot C^*(\mathbb{F})$ has a unique C*-norm. Since $J \subset C^*(\mathbb{F})$, is an ideal, the same holds for $B(\ell^2) \odot J$. Since the sequence
$$0 \to B(\ell^2) \otimes_{\max} J \to B(\ell^2) \otimes_{\max} C^*(F) \to B(\ell^2) \otimes_{\max} A \to 0$$
is also exact, we conclude that $B(\ell^2) \otimes_{\min} A = B(\ell^2) \otimes_{\max} A$.

On the other hand, if $B(\ell^2) \odot A$ has a unique C*-norm,then the sequence
$$0 \to B(\ell^2) \otimes_{\min} J \to B(\ell^2) \otimes_{\min} C^*(F) \to B(\ell^2) \otimes_{\min} A \to 0$$
is exact, which means id_A is locally liftable, which we have already shown implies A has the LLP. Either way, we have the following.

Theorem 14 (Kirchberg)**.** *A* C**-algebra A has the LLP if and only if $A \odot B(\ell^2)$ has a unique* C**-norm.*

In fact, a more general theorem holds.

Theorem 15. *Let A be a C^*-algebra. Then, TFAE*

1. *A has the LLP.*

2. $A \otimes_{\min} B(H) = A \otimes_{\max} B(H)$ *for any Hilbert space* H.

3. $A \otimes_{\min} B(\ell^2) = A \otimes_{\max} B(\ell^2)$.

Proof. We have already proved (3)⇔(1), so it remains to prove (1)⇒(2). The proof comes from [16].

Let $\mathbb{F}$ be a free group and J a two-sided ideal of $C^*(\mathbb{F})$ such that $A \simeq C^*(\mathbb{F})/J$, and let $\pi : C^*(\mathbb{F}) \to C^*(\mathbb{F})/J$ be the quotient map. For the sake of simplicity, assume $A = C^*(\mathbb{F})/J$. Let H be any Hilbert space, and $t = \sum_{i=1}^n a_i \otimes x_i \in (C^*(\mathbb{F})/J) \odot B(H)$. We will argue that

$$\|t\|_{(C^*(\mathbb{F})/J)\otimes_{\min} B(H)} \geq \|t\|_{(C^*(\mathbb{F})/J)\otimes_{\max} B(H)}$$

by considering a ucp map that maps $t \in (C^*(\mathbb{F})/J)\otimes_{\min} B(H)$ to $t \in (C^*(\mathbb{F})/J)\otimes_{\max} B(H)$. Let $E \subseteq C^*(\mathbb{F})/J$ be the operator system generated by $\{a_i\}_{i=1}^n$. Since $C^*(\mathbb{F})/J$ has the LLP, there is a ucp map $\phi : E \to C^*(\mathbb{F})$ such that $\pi \circ \phi(a) = a$ for all $a \in E$.

By the functoriality of $\otimes_{\max}$, we can extend $\pi \odot id_{B(H)}$ to a ucp map

$$\pi \otimes id_{B(H)} : C^*(\mathbb{F}) \otimes_{\max} B(H) \to (C^*(\mathbb{F})/J) \otimes_{\max} B(H).$$

Although we have so far only developed the theory of tensor products for C^*-algebras, we can also extend $\phi \odot id_{B(H)}$ to a ucp map

$$\phi \otimes id_{B(H)} : E \otimes_{\min} B(H) \to C^*(\mathbb{F}) \otimes_{\min} B(H).$$

where $E \otimes_{\min} B(H) \subseteq (C^*(\mathbb{F})/J) \otimes_{\min} B(H)$ as an operator system.[12]

Since $C^*(\mathbb{F}) \otimes_{\min} B(H) = C^*(\mathbb{F}) \otimes_{\max} B(H)$, we can compose the two ucp maps. Furthermore,

$$(\pi \otimes id_{B(H)}) \circ (\phi \otimes id_{B(H)})(t) = t.$$

Hence,

$$\|t\|_{(C^*(\mathbb{F})/J)\otimes_{\min} B(H)} = \|t\|_{E\otimes_{\min} B(H)} \geq \|t\|_{(C^*(\mathbb{F})/J)\otimes_{\max} B(H)}.$$

□

Example 8. *We may now give* $B(H)$ *(for any infinite dimensional Hilbert space* H*) as a non-example of a* C^**-algebra with the LLP because, as we saw in Example 3,*

$$B(H) \otimes_{\max} B(H) \neq B(H) \otimes_{\min} B(H).$$

Recall that $B(H)$ *has the WEP, so we now know that the WEP does not imply the LLP. In fact, we now know all of the following conjectures (plus a few more) are false:*

[12]Some natural concerns here are as follows: How is $\otimes_{\min}$ defined for operator systems? Is it functorial? Why is $E \otimes_{\min} B(H) \subseteq (C^*(\mathbb{F})/J) \otimes_{\min} B(H)$? Does this operator system $\otimes_{\min}$ agree with the C^*-algebra $\otimes_{\min}$ on C^*-algebras? These questions will be answered in the subsection of the appendix on tensor products for operator systems. For now, we take for granted that the structures are compatible as we claim.

Theorem 16 ([9]). *The following conjectures are equivalent:*

1. *$Ext(A)$ is a group for every separable unital C^*-algebra A with the WEP.*
2. *Every finite dimensional operator system is unitally completely isometrically isomorphic to an operators system in $C^*(\mathbb{F}_\infty)$.*
3. *For every pair A and B of separable unital C^*-algebras with the WEP,*

$$A \otimes_{\max} B = A \otimes_{\min} B.$$

4. *The WEP and approximate injectivity are equivalent. (See Remark 1.)*

5 Main Theorem

We have almost finished proving the tensorial duality for the LLP and WEP. Once established, this will take care of most of our goal theorem.

Theorem 17 (Kirchberg,[9]). *Let A and B be C^*-algebras. Then*

1. *A has the LLP $\Leftrightarrow A \otimes_{\max} B(\ell^2) = A \otimes_{\min} B(\ell^2)$,*
2. *B has the WEP $\Leftrightarrow C^*(\mathbb{F}_\infty) \otimes_{\max} B = C^*(\mathbb{F}_\infty) \otimes_{\min} B$, and*
3. *A has the LLP and B has the WEP $\Rightarrow A \otimes_{\max} B = A \otimes_{\min} B$.*

Proof. It remains to show (3). To that end, faithfully embed $B \subseteq B(H)$ for some Hilbert space H. Since B has the WEP, $A \otimes_{\max} B \subseteq A \otimes_{\max} B(H)$; since A has the LLP, $A \otimes_{\max} B(H) = A \otimes_{\min} B(H)$. Therefore, we have

$$A \otimes_{\max} B \subseteq A \otimes_{\max} B(H) = A \otimes_{\min} B(H) \supseteq A \otimes_{\min} B.$$

In other words, the topologies (and hence norms) agree. □

Remark 8. *Notice that both $B(\ell^2)$ and $C^*(\mathbb{F}_\infty)$ are universal among separable C^*-algebras in that any separable C^*-algebra embeds into one and is a quotient of the other.*

Now, recall our goal theorem:

Theorem 18. *The following are equivalent:*

1. *$C^*(\mathbb{F}_\infty) \otimes_{max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{min} C^*(\mathbb{F}_\infty)$, where $\mathbb{F}_\infty$ is the free group on countably many generators. (Kirchberg's Conjecture)*
2. *$C^*(\mathbb{F}_\infty)$ has the WEP.*
3. *All C^*-algebras are QWEP. (QWEP Conjecture)*
4. *LLP $\Rightarrow$ WEP.*

Proof. First, we will handle the implications that are immediate consequences of Theorem 17.

(1)⇔(2) This follows immediately from Theorem 17 (2).

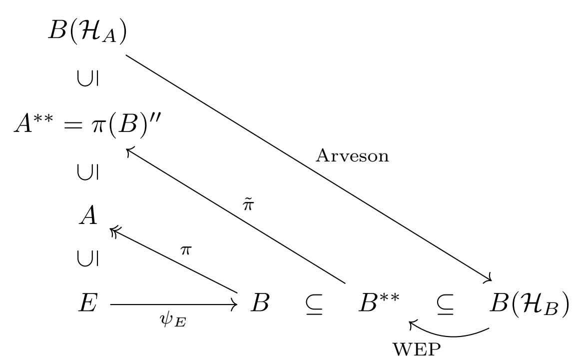
(4)⇒(1) This follows from Theorem 17(3) and the fact that $C^*(\mathbb{F}_\infty)$ has the LLP.

(2)⇒(3) Since $C^*(\mathbb{F})$ has the LLP for any free group $\mathbb{F}$, by Theorem 17(3), if $C^*(\mathbb{F}_\infty)$ has the WEP, then

$$C^*(\mathbb{F}) \otimes_{max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}) \otimes_{min} C^*(\mathbb{F}_\infty)$$

for any free group $\mathbb{F}$. But by Theorem 17(2), this is equivalent to $C^*(\mathbb{F})$ having the WEP. Since any C^*-algebra can be identified with the quotient of $C^*(\mathbb{F})$ for some free group $\mathbb{F}$, we have that any C^*-algebra is QWEP.

(3)⇒(4) (This proof follows [14].) It will suffice to prove that QWEP + LLP ⇒ WEP. To that end, suppose $A \subseteq B(\mathcal{H}_A)$ has the LLP and is QWEP, and let B be a C^*-algebra with the WEP and $\pi : B \to A$ a surjective *-homomorphism. Without loss of generality, we may assume A and B are unital or replace them with their unitizations. For a finite dimensional operator system $E \subseteq A$, let $\psi_E : E \to B$ denote the ucp lifting of $id_A|_E$. Let $\tilde{\pi} : B^{**} \to A^{**}$ be the normal extension of π. Then Arveson's theorem (for $E \subseteq B(\mathcal{H}_A)$ and $\phi_E : E \to B(\mathcal{H}_B)$) along with the WEP give us a ucp map $\phi_E : B(\mathcal{H}_A) \to A^{**}$ such that $\phi_E|_E = id_E$.



Let $\phi : B(\mathcal{H}_A) \to A^{**}$ be any cluster point of the net of ucp maps $\{\phi_E\}$ in the pointwise weak*-topology. Then ϕ is our weak conditional expectation.

□

6 Connections to CEP

A little more intermediate theory is required to show that a positive answer to Kirchberg's conjecture (1) would imply a positive answer to Connes Embedding

Problem (1), and we refer the reader to section 13.3 (and the prerequisite 6.2) in [2]. However, if one will accept a few permanence properties of the WEP and QWEP, we can explain why a positive answer to CEP would imply that the QWEP conjecture is true.

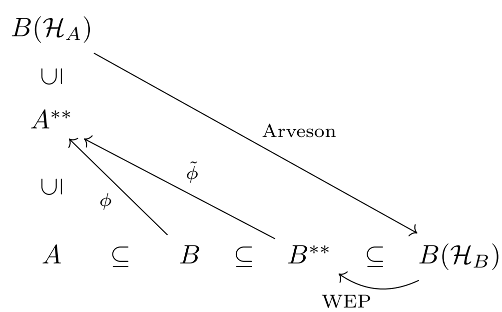
6.1 Some Permanence Properties of WEP and QWEP

Here we prove or just list some relevant permanence properties of the WEP and QWEP. For more, see [9]. The most utilized permanence property is the following:

Proposition 9. *Suppose $A \subseteq B$ are C^*-algebras, B is (Q)WEP, and there exists a weak conditional expectation from B to A^{**}. Then A is (Q)WEP.*

Since the intermediate lemmas that establish this result, we will give them and argue for what is not explicitly proved in the sources. First, we sketch the proof of the claim for WEP:

Proof for WEP case. Suppose $A \subseteq B$, B is WEP, and $\phi : B \to A^{**}$ is a weak conditional expectation. From Banach space theory, we know that the bounded linear map $\phi : B \to (A^*)^*$ uniquely extends to a weak*-continuous map $\tilde{\phi} : B^{**} \to A^{**}$, which is actually a conditional expectation. We embed $A \subseteq A^{**} \subseteq B(H_A)$ and $B \subseteq B^{**} \subseteq B(H_B)$ (where $(\pi_A, B(H_A))$ and $(\pi_B, B(H_B))$ are the respective universal representations). Being fast and loose with our "$\subset$" symbol gives us the following diagram:



Following the diagram gives our desired weak conditional expectation $B(H_A) \to A^{**}$. □

Since $id_{A^{**}}$ is a weak conditional expectation for $A \subseteq A^{**}$, we get the following corollary:

Corollary 4. *If A^{**} is WEP, then so is A.*

Now, to prove the proposition for the case of B QWEP. This argument is an embellishment of the one given in [2].

Proof. Suppose $A \subseteq B$, B is QWEP, and $\phi : B \to A^{**}$ is a weak conditional expectation. Again, let $\tilde{\phi} : B^{**} \to A^{**}$ be our extension. Suppose C is a C^*-algebra with the WEP with *-homomorphism $\pi : C \to B$. Then we get a normal extension $\pi : C^{**} \to B^{**}$ with

$$\pi^{-1}(A)^{**}) \simeq (ker\pi)^{**} \oplus A^{**} \subseteq (ker\pi)^{**} \oplus B^{**} \simeq C^{**}.$$

So, we have a ucp map

$$\psi = id_{(ker\pi)^{**}} \oplus \tilde{\phi} : C^{**} \to \pi^{-1}(A)^{**}$$

such that $\psi|_{\pi^{-1}(A)} = id_{\pi^{-1}(A)}$. Then, $\psi|_C : C \to \pi^{-1}(A)^{**}$ is a weak conditional expectation. Since C is WEP, this implies that $\pi^{-1}(A)$ is also WEP and A is a quotient of $\pi^{-1}(A)$. □

Again, since $id_{A^{**}}$ is a weak conditional expectation for $A \subseteq A^{**}$, we have the following corollary.

Corollary 5. *A C^*-algebra A is QWEP if A^{**} is QWEP.*[13]

We conclude the section with a few properties that we will not prove.

Proposition 10. *Let A_i below be C^*-algebras*

1. *WEP is preserved by complete isomorphisms. (see [16] 15.10)*
2. *A_i is (Q)WEP for every $i \in \mathcal{I}$ iff $\prod_{i \in \mathcal{I}} A_i$ is (Q)WEP.*

   (see [9] 3.3)
3. *Let $\{A_i\}_{i \in \mathcal{I}}$ be an increasing net of C^*-subalgebras in $B(H)$. If A_i is (Q)WEP for every $i \in \mathcal{I}$, then so is $\overline{\bigcup A_i}$. (see [2] 13.3.6)*

6.2 CEP ⇒ QWEP

Here we briefly sketch the argument (from [2]) for why a positive answer to Connes Embedding Conjecture (1) would imply a positive answer to the QWEP conjecture. The argument has two main components:

1. Assume CEP and show that every finite von Neumann algebra with separable predual is QWEP.
2. Justify why every finite von Neumann algebra with spearable predual being QWEP implies that every C^*-algebra is QWEP.

Proof. The proofs of the two components go roughly as follows:

[13] iff, see [9] or [14]

1. Note that we have already used the second property in Prop 10 to argue that an ultrapower of the hyperfinite II_1-factor is QWEP. Since every von Neumann subalgebra of a finite von Neumann algebra[14] is the range of a conditional (and hence weak conditional) expectation, any von Neumann subalgebra of R^ω is QWEP by Prop 9. By assumption, this means that all finite von Neumann algebras with separable predual are QWEP.

2. A consequence of the third property in Prop 10 is that, if every finite von Neumann algebra with separable predual is QWEP, then every semifinite von Neumann Algebra is QWEP. From Takesaki's work in modular theory, we know that any von Neumann algebra can be realized as the image of a conditional expectation from a semifinite von Neumann algebra (see [2] 9.3.5-7); hence if all semifinite von Neumann algebras are QWEP, then all von Neumann algebras are QWEP. This implies that all double-duals of C^*-algebras are QWEP, which in turn implies that all C^*-algebras are QWEP.

□

7 Appendix

7.1 Kirchberg's Theorem

This theorem was far too important to our proofs to not offer a proof. However, time was constrained, so we hope that adding the proof to this document will atone for its absence in the talks.

Theorem 19 (Kirchberg)**.** *For any free group $\mathbb{F}$ and any Hilbert space H,*

$$C^*(\mathbb{F}) \otimes_{max} B(H) = C^*(\mathbb{F}) \otimes_{min} B(H).$$

Remark 9. *Kirchberg's original proof is for a free group on countably many generators. Pisier's argument allows us to immediately start working with general free groups in our proofs.*

The following proof is due to Pisier, [16], but we will follow the proof in [2].

Proof. By similar arguments to those above, it will suffice to prove the claim for consider $\mathbb{F}_{n-1}$ for some $n \geq 3$. We shall prove the claim for $H = \ell^2$, the more general case will follow in similar fashion.

Let E_n be the n-dimensional operator space in $C^*(\mathbb{F}_{n-1})$ spanned by $1 = U_0$ and the unitary generators U_k of $C^*(\mathbb{F}_{n-1})$. This operator space is the universal operator space generated by n contractions with respect to cc maps[15] Since

[14] R is finite.

[15] i.e. Given n contractions $a_0, ..., a_n \in B(H)$, there exists a cc map $\theta : E_n \to B(H)$ mapping $U_k \mapsto a_k$. This follows from the universality of $C^*(\mathbb{F}_{n-1})$ and unitary dilations of the a_k. For the argument, see the proof of 13.2.2 in [2].

$C^*(\mathbb{F}_{n-1})$ is the universal C^*-algebra generated by $n-1$ unitaries, by looking at suitable representations, we see that for any $(\alpha_k)_{k=0}^{n-1} \in \mathbb{C}^n$,

$$\left\| \sum_{k=0}^{n-1} \alpha_k U_k \right\| = \sum_{k=0}^{n-1} |\alpha_k|.$$

Hence, we see that E_n is canonically isometric to ℓ_n^1. By duality, we have a one-to-one correspondence between elements $z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes_{min} B(\ell^2)$[16] and maps $T_z : \ell_n^\infty \to B(\ell^2)$ given by

$$(\alpha_k) \mapsto \sum \alpha_k x_k.$$

Lemma 3. *E_n is canonically completely isometrically isomorphic to the dual operator space $\ell_n^1 = (\ell_n^\infty)^*$, or, equivalently, $\|z\|_{min} = \|T_z\|_{cb}$ for every $z \in E_n \otimes B(\ell^2)$.*

Proof. Since $(U_k)_{k=0}^{n-1} \in E_n \otimes \ell_n^\infty$ is contractive and $z \in E_n \otimes_{min} B(\ell^2)$ has the form $(id_{E_n} \otimes T_z)((U_k))$, we have that $\|z\|_{min} \leq \|T_z\|_{cb}$. On the other hand, let $a_0, ..., a_{n-1} \in B(H)$ for some Hilbert space H, and let $\theta : E_n \to B(H)$ be the cc map mapping $U_k \mapsto a_k$ guaranteed by universality. Then, for $(a_k)_{k=0}^{n-1} \in B(H) \otimes \ell_n^\infty$ and any $z = \sum U_k \otimes x_k \in E_n \otimes_{min} B(\ell^2)$,

$$\|(id_{B(H)} \otimes T_z)((a_k)_{k=0}^{n-1}\|_{min} = \| \sum_{k=0}^{n-1} a_k \otimes x_k \|_{min} = \|(\theta \otimes id_{B(\ell^2)})(z)\|_{min} \leq \|z\|_{min}.$$

Since $(a_k)_{k=0}^{n-1} \in B(H) \otimes \ell_n^\infty$ was arbitrary, $\|T_z\|_{cb} \leq \|z\|_{min}$. □

Lemma 4. *Let $X_i \subseteq B(H_i)$ $(i = 1, 2)$ be unital operator subspaces and let $\phi : X_1 \to X_2$ be a unital complete isometry. Suppose that $\phi(X_1)$ is spanned by unitary elements in $B(H_2)$. Then ϕ uniquely extends to a $*-$ homomorphism between the C^*-subalgebras $C^*(X_i)$ generated by X_i in $B(H_i)$.*

Proof. By Arveson's Extension Theorem (for unital operator spaces), ϕ extends to a ucp map from $B(H_1)$ to $B(H_2)$, also denoted ϕ. Since $\phi|_{X_1}$ is unital and isometric and since X_2 is spanned by unitaries, X_1 is contained in the multiplicative domain[17] of ϕ, i.e. $\phi|_{X_1}$ is a *-homomorphism, and so ϕ is a $*$-homomorphism on $C^*(X_1)$. □

By the second lemma, it suffices to show that the formal identity map $E_n \otimes_{min} B(\ell^2) \to C^*(\mathbb{F}_{n-1}) \otimes_{max} B(\ell^2)$ is cc for every n. To that end, let $z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes_{min} B(\ell^2)$ with $\|z\|_{min} = 1$. By the first lemma, the map $T_z : \ell_n^\infty \to B(\ell^2)$ is cc. By the factorization theorem for cb maps[18], there

[16]The min norm for operator spaces is also induced by the embeddings.

[17]The multiplicative domain of ϕ is
$\{x \in B(H_1) : \phi(xy) = \phi(x)\phi(y)\ \&\ \phi(yx) = \phi(y)\phi(x)\ \forall\ y \in B \otimes_{max} C\}$,
or equivalently $\{x \in B(H_1) : \phi(xx^*) = \phi(x)\phi(x^*)\ \&\ \phi(x^*x) = \phi(x)^*\phi(x)\}$,
or equivalently the largest subalgebra $A \subseteq B(H_1)$ such that $\phi|_A$ is a $*$-homomorphism.

[18]The factorization theorem for cb maps:

exists a Hilbert space H, a *-homomorphism $\pi : \ell_n^\infty \to B(H)$, and isometries $V, W \in B(\ell^2, H)$ such that $T_z(f) = V^*\pi(f)W$ for all $f \in \ell_n^\infty$; we may assume $H = \ell^2$. Let $a_k := \pi(\delta_k)V$ and $b_k := \pi(\delta_k)W$ in $B(\ell^2)$. Then, $x_k := a_k^* b_k$ for each k, and $\sum_{k=0}^{n-1} a_k^* a_k = 1 = \sum_{k=0}^{n-1} B_k^* b_k$. Hence,

$$\begin{aligned}\left\|\sum_{k=0}^{n-1} U_k \otimes x_k\right\|_{C^*(\mathbb{F}_{n-1})\otimes_{max} B(\ell^2)} &= \left\|\sum_{k=0}^{n-1}(1\otimes a_k)^*(U_k \otimes b_k)\right\|_{max} \\ &\leq \left\|\sum_{k=0}^{n-1}(1\otimes a_k)^*(1\otimes a_k)\right\|_{max}^{1/2} \left\|\sum_{k=0}^{n-1}(U_k \otimes b_k^*(U_k \otimes b_k)\right\|_{max}^{1/2} \\ &\leq 1,\end{aligned}$$

which implies that the formal identity map $E_n \otimes_{min} B(\ell^2) \to C^*(\mathbb{F}_{n-1}) \otimes_{max} B(\ell^2)$ is contractive. Since $B(\ell^2) \simeq M_m(B(\ell^2))$ for all $m \geq 1$, this implies that the map is actually completely contractive. □

7.2 Tensor Products for Operator Systems

For the LLP, it behoves us to consider how to define an operator system structure on the tensor products operator systems. For a resource, we direct you to Kavruk, Paulsen, Todorov, and Tomforde's development these concepts in [8]. The theory guarantees that they play nicely and intuitively with the same notions for C^*-algebras. For the sake of simplicity, we give pertinent characterizations for definitions.

Definition 12. *For two operator systems S_i with unital completely isometric embeddings $\iota_i : S_i \to B(H_i)$, $S_1 \otimes S_2$ is the operator structure arising from the embedding $\iota_1 \odot \iota_2 : S_1 \odot S_2 \to B(H_1 \otimes H_2)$.*

Since the restriction of a faithful $*$-homomorphism on a C^*-algebra A to an operator system S contained in A is uci, we can conclude that for C^*-algebras A_i with sub-operator systems $S_i \subseteq A_i$, there is a natural embedding $S_1 \otimes_{min} S_2 \subseteq A_1 \otimes_{min} A_2$. Hence, for $x \in S_1 \odot S_2$, we can say $\|x\|_{S_1 \otimes_{min} S_2} = \|x\|_{A_1 \otimes_{min} A_2}$.

Furthermore, if A and B are C^*-algebras, then their operator system tensor product $A \otimes_{\min} B$ agrees with their C^*-tensor product $A \otimes_{\min} B$. (Hence, in the proof in the preceding section, we were indeed justified in saying that we could compose two ucp maps $(\pi \otimes id_{B(H)})$ and $(\phi \otimes id_{B(H)})$.)

Furthermore, the operator system tensor product $\otimes_{min}$ is functorial, i.e.

Theorem 20 (Haagerup, Paulsen, Wittstock)**.** *Let $X \subseteq A$ be an operator space and $\phi : X \to B(H)$ a cc map. Then there exists a Hilbert space K, a *-rep $\pi : A \to B(K)$, and isometries $V, W : H \to K$ such that*

$$\phi(x) = V^*\pi(x)W$$

for every $x \in X$. In particular, ϕ extends to a cc map on A.

Proposition 11. *[8] Given ucp maps $\phi_i : S_i \to T_i$ between operator systems, the map*

$$\phi_1 \odot \phi_2 : S_1 \odot S_2 \to T_1 \odot T_2$$

extends to a ucp map

$$\phi_1 \otimes_{min} \phi_2 : S_1 \otimes_{min} S_2 \to T_1 \otimes_{min} T_2$$

7.3 Residual Finite Dimensionality

Another common item in the list for Theorem 1 is that $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty)$ is residually finite dimensional or RFD (i.e. it has a separating family of finite-dimensional representations). In this section, we offer a proof for why this is equivalent to Kirchberg's Conjecture, i.e. that

$$C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) \;\Leftrightarrow\; C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty) \text{ is RFD.}$$

We first note that $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty) \simeq C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$ (which follows from the universality of the full group C^*-algebra and that of $\otimes_{\max}$). Furthermore, Choi proved in [4] that $C^*(\mathbb{F}_\infty)$ is RFD. Hence, equivalence of the two conjectures follows from the following proposition.

Proposition 12. *If A and B are RFD unital C^*-algebras, then $A \otimes B$ has a unique C^*-norm iff $A \otimes_{max} B$ is RFD.*

Proof. The ($\Rightarrow$) claim follows immediately from the fact that $A \otimes_{min} B$ is RFD when A and B are RFD. (Indeed, if $A \otimes_{\min} B$ is RFD, then so are A and B as subalgebras; on the other hand, if A and B are RFD, with separating families of f.d. representations $\{\pi_\alpha\}$ and $\{\sigma_\beta\}$, then $\pi = \oplus_\alpha \pi_\alpha$ and $\sigma = \oplus_\beta \sigma_\beta$ are faithful representations. Hence, $\{\pi_\alpha \otimes \sigma_\beta\}_{\alpha,\beta}$ yields a separating family of finite dimensional representations for $A \otimes_{\min} B$.)

For the ($\Leftarrow$) first note that, since both the injective and projective tensor products are RFD, both norms are well-approximated by finite dimensional representations. So, to show that the two norms agree, it suffices to show that for any finite dimensional representation $\pi : A \otimes_{max} B \to M_n$, we can find a representation $\pi' : A \otimes_{min} B \to M_n$ so that

$$\left\| \sum_{i=1}^{n} \pi'(a_i) \otimes \pi'(b_i) \right\| \geq \left\| \sum_{i=1}^{n} \pi(a_i \otimes b_i) \right\|.$$

In fact, given any finite-dimensional representation $\pi : A \otimes_{max} B \to M_n$, we claim that $\pi|_{A \odot B}$ extends to a representation of σ $A \otimes_{min} B$ into M_n, i.e. π factors through $A \otimes_{min} B$ via the quotient map and a representation σ of $A \otimes_{min} B$:

$$\begin{array}{ccccc} A \otimes_{max} B & & \xrightarrow{\quad\pi\quad} & & M_n \\ & \searrow^{q} & & \nearrow^{\sigma} & \\ & & A \otimes_{min} B & & \end{array}$$

To that end, let $\pi : A \otimes_{max} B \to M_n$ be a finite dimensional representation and define the representations

$$\pi_A : A \to M_n \text{ by } \pi_A(a) = \pi(a \otimes 1)$$
$$\pi_B : B \to M_n \text{ by } \pi_B(b) = \pi(1 \otimes b).$$

Then, $\pi_A(A)$ and $\pi_B(B)$ are commuting C^*-algebras. Hence, the natural inclusions $\iota_A : \pi_A(A) \to M_n$ and $\iota_B : \pi_B(B) \to M_n$ are representations with commuting ranges. By the universality of $\otimes_{max}$ there is a unique representation

$$(\iota_A \times \iota_B) : \pi_A(A) \otimes_{max} \pi_B(B) \to M_n$$

with $\pi_A(a) \otimes \pi_B(b) \mapsto \pi_A(a)\pi_B(b)$ for all $a \in A$ and $b \in B$.

On the other hand, let

$$\pi_A \otimes \pi_B : A \otimes_{min} B \to \pi_A(A) \otimes_{min} \pi_B(B)$$

be the representation induced by π_A and π_B, i.e. for all $a \in A$ and $b \in B$

$$(\pi_A \otimes \pi_B)(a \otimes b) = \pi_A(a) \otimes \pi_B(b).$$

Now, since $\pi_A(A)$ and $\pi_B(B)$ are finite-dimensional C^*-algebras, they are nuclear, i.e. $\pi_A(A) \otimes_{max} \pi_B(B) = \pi_A(A) \otimes_{min} \pi_B(B)$. Thus, we may define the representation

$$\sigma := (\iota_A \times \iota_B) \circ (\pi_A \otimes \pi_B) : A \otimes_{min} B \to M_n$$

of $A \otimes_{min} B$ where for $a \otimes b \in A \odot B$,

$$\sigma(a \otimes b) = (\iota_A \times \iota_B)(\pi_A(a) \otimes \pi_B(b)) = \pi_A(a)\pi_B(b) = \pi(a \otimes 1)\pi(1 \otimes b) = \pi(a \otimes b).$$

Then, σ is an extension of $\pi|_{A \odot B}$. □

What we have shown is that a finite-dimensional representation of $A \otimes_{max} B$ factors through $A \otimes_{min} B$, i.e. the following diagram commutes

$$\begin{array}{ccc} A \otimes_{max} B & \xrightarrow{\quad\pi\quad} & M_n \\ \phi \downarrow & & \uparrow \iota_A \times \iota_B \\ A \otimes_{min} B & \xrightarrow{\pi_A \otimes \pi_B} & \pi_A(A) \otimes \pi_B(B) \end{array}$$

where ϕ is the quotient map; or, more simply, this diagram commutes:

$$\begin{array}{ccc} A \otimes_{max} B & \xrightarrow{\quad\pi\quad} & M_n \\ {\scriptstyle q}\searrow & & \nearrow{\scriptstyle \sigma} \\ & A \otimes_{min} B & \end{array}$$

where q is the quotient map.

References

[1] Brown, Nathanial P., Invariant Means and Finite Representation Theory of C^*-algebras, *Memoirs of the American Mathematical Society* **184** (2006), no. 865, vi+105pp.

[2] Brown, Nathanial P., Ozawa, Narutaka. *C^*-Algebras and Finite-Dimensional Approximations*. Graduate Studies in Mathematics. Volume 88 AMS. (2008).

[3] Connes, A.. Classification of Injective Factors. Cases $II_1, II_\infty, III_\lambda, \lambda \neq 1$. *Ann. Math.* (2) **104** (1976), no. 1, 73-115.

[4] Choi, Man-Duen. The Full C^*-Algebra of the Free Group on Two Generators. *Pacific Journal of Mathematics*. Vol. 87, No. 1 (1980).

[5] Davidson, Kenneth. *C^*-algebras By Example*. Fields Institute Monographs, Amer. Math. Soc., Providence, R.I., 1996.

[6] Effros, Edward G., Haagerup, Uffe. Lifting problems and local reflexivity for C^* -algebras. *Duke Math. J.* Col. 52 (1985), no. 1, 103-128.

[7] M. Junge, G. Pisier. *Bilinear Forms on Exact Operator Spaces and $B(H) \otimes B(H)$*, Geom. Funct. Anal. 5 (1995), 329-363.

[8] A. Kavruk, V.I. Paulsen, I.G. Todorov and M. Tomforde. Tensor products of operator systems. *J. of Functional Analysis* Vol. 261, Issue 2, (2011), 267-299.

[9] Kirchberg, Eberhard. On non-semisplit extensions, tensor products and exactness of group C^*-algebras. Springer-Verlag. *Invent. math.* 112, 449-489 (1993).

[10] Kirchberg, Eberhard. Commutants of unitaries in UHF algebras and functorial properties of exactness. *J. Reine Angew. Math.* 452 (1994), 39-77.

[11] Kirchberg, Eberhard. Discrete groups with Kazhdan's property T and factorization property are residually finite. *Math. Ann.* 299 (1994), no. 3, 551-563.

[12] C. Lance. On nuclear C^*-algebras. *J. Functional Analysis* 12 (1973), 157-176.

[13] Loring, Terry. *Lifting Solutions to Perturbing Problems in C^*-Algebras.* Fields Institutue Monographs, 8. AMS, Providence, RI (1997).

[14] Ozawa, Narutaka. About the QWEP Conjecture. *Int. J. Math.*, 15, 501 (2004).

[15] Ozawa, Narutaka. There is no separable universal II_1-factor. *Proc. Amer. Math. Soc.* 132 (2004), no. 2, 487-490 (electronic).

[16] Pisier, Gilles. *Introduction to Operator Space Theory.* London Mathematical Society Lecture Note Series, 294. Cambridge University Press (2003).

[17] M. Takesaki, *Theory of Operator Algebras, I.* Springer-Verlag, New York, 1979.

[18] A. Thom. Examples of Hyperlinear Groups Without Factorization Property. *Groups Geom. Dyn.* 4(1), 195-208 (2010).