Kirchberg's QWEP Conjecture: Between Connes' and Tsirelson's Problems

Kristin Courtney
WWU Münster

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$MIP^* = RE$

Early this year, a negative answer to the Connes Embedding Problem was announced by Ji, Natarajan, Vidick, Wright and Yuen in their paper $MIP^* = RE$.

Question (Connes, 1976)

Does every separably acting type II_1-factor embed into some ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$?

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Actually, the authors use quantum complexity theory to give a negative answer to Tsirelson's problem from quantum information theory.

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Theorem (Fritz, Junge et al, Ozawa)

Tsirelson's Problem has a positive solution if and only if for each $k, m \geq 2$ with either $k > 2$ or $m > 2$,

$$\mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*k}) = \mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*k}).$$

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Many of the results and arguments therein (as well as some of Kirchberg's peripheral work) have been clarified and augmented by various authors in the ensuing years. In this talk, we rely heavily on expositions and improvements from Pisier, Ozawa, and Brown-Ozawa.

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The story begins with injectivity.

Injectivity

A unital C^*-algebra A is *injective* if for any embedding $A \subset B$, there exists a ucp projection $\psi : B \to A$.

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Theorem (Arveson)

Let $A \subset B$ be C^*-algebras and $\mathcal{H}$ a Hilbert space. Any ucp map $\phi : A \to B(\mathcal{H})$ has a ucp extension $\psi : B \to B(\mathcal{H})$.

Weak Injectivity

Suppose instead that for some embedding $A \subset B$, there exists a ucp map $\psi : B \to A^{**}$, which extends the canonical embedding $A \hookrightarrow A^{**}$.

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To show that A has the WEP, it suffices to check this on the embedding $\pi_u : A \to B(\mathcal{H}_A)$.

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All injective C^*-algebras have WEP, including the hyperfinite II_1-factor

$$\mathcal{R} = \overline{\bigotimes_{\mathbb{N}}} \, \mathbb{M}_2.$$

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Corollary (to Takesaki's Modular Theory)

For any von Neumann algebra M, there is a semifinite von Neumann algebra N so that M embeds into N as the range of a conditional expectation.
(In particular, $N = M \rtimes_\alpha \mathbb{R}$ where α is the modular action.)

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Theorem (Kirchberg)

CEP $\Rightarrow$ *Kirchberg's QWEP Conjecture.*

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For our argument, we reduce to the case where $\mathrm{C}^*(\mathbb{F})$ is separable. The results are the same in the non-separable case, but the proofs are more involved.

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It turns out this is enough to imply that all separable free group C^*-algebras have WEP.

This is because, $\mathrm{C}^*(\mathbb{F})$ has the “dual” property to the WEP: the *lifting property*.

From QWEP to WEP

Theorem (Kirchberg)

Let $\mathbb{F}$ be a free group on countably many generators. Then for any C^*-algebras A and B with a surjective $*$-homomorphism $\pi : B \to A$, any ucp map $\phi : \mathrm{C}^*(\mathbb{F}) \to A$ lifts to a ucp map $\psi : \mathrm{C}^*(\mathbb{F}) \to B$ such that $\pi\psi = \phi$.

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ \mathrm{C}^*(\mathbb{F}) \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"] & A \end{tikzcd}$$

From QWEP to WEP

$$
\begin{array}{c}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \\
\cup \\
\pi_u(\mathrm{C}^*(\mathbb{F}))'' \\
\cup \\
\pi_u(\mathrm{C}^*(\mathbb{F})) \\
{\scriptstyle \pi_u}\big\uparrow \\
\mathrm{C}^*(\mathbb{F})
\end{array}
$$

From QWEP to WEP

$$\begin{array}{c} B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \\ \cup \\ \mathrm{C}^*(\mathbb{F})^{**} \\ \cup \\ \mathrm{C}^*(\mathbb{F}) \\ \Big\| \\ \mathrm{C}^*(\mathbb{F}) \end{array}$$

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$$
\begin{array}{c}
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\cup \\
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\Big\| \\
\mathrm{C}^*(\mathbb{F})
\end{array}
$$

From QWEP to WEP

$$\begin{array}{ccc} B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) & & \\ \cup & & \\ \mathrm{C}^*(\mathbb{F})^{**} & & \\ \cup & & \\ \mathrm{C}^*(\mathbb{F}) & & \\ \| & \overset{\pi}{\nwarrow} & \\ \mathrm{C}^*(\mathbb{F}) & & B \end{array}$$

Suppose there is a C^*-algebra B with the WEP and a surjection $\pi : B \to \mathrm{C}^*(\mathbb{F})$.

From QWEP to WEP

$$
\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \arrow[d, phantom, "\cup"] & \\
\mathrm{C}^*(\mathbb{F})^{**} \arrow[d, phantom, "\cup"] & \\
\mathrm{C}^*(\mathbb{F}) \arrow[d, equal] & \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, dashed, "LP"'] & B \arrow[ul, two heads, "\pi"']
\end{tikzcd}
$$

By Kirchberg's theorem, there exists a ucp map $\mathrm{C}^*(\mathbb{F}) \to B$ so that the above diagram commutes.

From QWEP to WEP

$$\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \\
\cup \\
\mathrm{C}^*(\mathbb{F})^{**} \\
\cup \\
\mathrm{C}^*(\mathbb{F}) \arrow[d, equal] \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, "LP"'] & B \arrow[ul, two heads, "\pi"'] & \subset & B(\mathcal{H}_B)
\end{tikzcd}$$

Now, we identify B with its image in its universal representation $B(\mathcal{H}_B)$.

From QWEP to WEP

$$
\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) & \\
\cup & \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, "ucp"'] & B(\mathcal{H}_B)
\end{tikzcd}
$$

From QWEP to WEP

$$\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \arrow[drr, "\text{Arveson}"] & & \\
\cup & & \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, "ucp"'] & & B(\mathcal{H}_B)
\end{tikzcd}$$

Arveson's theorem allows us to extend the map $\mathrm{C}^*(\mathbb{F}) \to B(\mathcal{H}_B)$ to a ucp map $B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \to B(\mathcal{H}_B)$.

From QWEP to WEP

$$\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \arrow[d, phantom, "\cup"] \arrow[dddrr, "\text{Arveson}"] & & \\
\mathrm{C}^*(\mathbb{F})^{**} \arrow[d, phantom, "\cup"] & & \\
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Arveson's theorem allows us to extend the map $\mathrm{C}^*(\mathbb{F}) \to B(\mathcal{H}_B)$ to a ucp map $B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \to B(\mathcal{H}_B)$.

From QWEP to WEP

$$
\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \arrow[dddddrrrrr, "\text{Arveson}"] & & & & & \\
\cup & & & & & \\
\mathrm{C}^*(\mathbb{F})^{**} & & & & & \\
\cup & & & & & \\
\mathrm{C}^*(\mathbb{F}) \arrow[d, equal] & & & & & \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, "LP"'] & B \arrow[ul, two heads, "\pi"'] & \subset & B^{**} & \subset & B(\mathcal{H}_B)
\end{tikzcd}
$$

Inside $B(\mathcal{H}_B)$, B sits inside its double commutant, which we write as B^{**}.

From QWEP to WEP

$$
\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \arrow[d, phantom, "\cup"] \arrow[dddrrrrr, "\text{Arveson}"] & & & & & \\
\mathrm{C}^*(\mathbb{F})^{**} \arrow[d, phantom, "\cup"] & & & & & \\
\mathrm{C}^*(\mathbb{F}) \arrow[d, equal] & & & & & \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, "LP"'] & B \arrow[ul, two heads, "\pi"'] & \subset & B^{**} & \subset & B(\mathcal{H}_B) \arrow[ll, bend left, "\text{WEP}"]
\end{tikzcd}
$$

Since B has the WEP, there exists a ucp map $B(\mathcal{H}_B) \to B^{**}$, which restricts to the identity on B.

From QWEP to WEP

$$
\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \arrow[d, phantom, "\cup"] \arrow[rrrddd, "\text{Arveson}"] \\
\mathrm{C}^*(\mathbb{F})^{**} \arrow[d, phantom, "\cup"] \\
\mathrm{C}^*(\mathbb{F}) \arrow[d, equal] \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, "LP"'] & B \arrow[ul, two heads, "\pi"'] \arrow[r, phantom, "\subset"] & B^{**} \arrow[uull, "\pi^{**}"'] \arrow[r, phantom, "\subset"] & B(\mathcal{H}_B) \arrow[l, bend left, "\text{WEP}"]
\end{tikzcd}
$$

Let $\pi^{**} : B^{**} \to \mathrm{C}^*(\mathbb{F})^{**}$ be the normal extension of $\pi : B \to \mathrm{C}^*(\mathbb{F})$.

From QWEP to WEP

$$\begin{tikzcd}
B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \arrow[dddddrrrrr, "\text{Arveson}"] & & & & & \\
\cup & & & & & \\
\mathrm{C}^*(\mathbb{F})^{**} & & & & & \\
\cup & & & & & \\
\mathrm{C}^*(\mathbb{F}) \arrow[d, equal] & & & & & \\
\mathrm{C}^*(\mathbb{F}) \arrow[r, "LP"'] & B \arrow[ul, two heads, "\pi"'] & \subset & B^{**} \arrow[uuulll, "\pi^{**}"'] & \subset & B(\mathcal{H}_B) \arrow[ll, bend left, "\text{WEP}"]
\end{tikzcd}$$

And thus we have our desired ucp map $B(\mathcal{H}_{\mathrm{C}^*(\mathbb{F})}) \to \mathrm{C}^*(\mathbb{F})^{**}$ that restricts to the identity on $\mathrm{C}^*(\mathbb{F})$.

Scorecard

Consider the following:

1. Connes' Embedding Problem
2. Every C^*-algebra is QWEP.
3. $\mathrm{C}^*(\mathbb{F})$ has WEP for any (every) non-abelian free group $\mathbb{F}$.
4. $\mathrm{C}^*(\mathbb{F}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}) = \mathrm{C}^*(\mathbb{F}) \otimes_{\min} \mathrm{C}^*(\mathbb{F})$ for any (every) non-abelian free group $\mathbb{F}$.
5. $\mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*k}) = \mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*k})$ for each $k, m \geq 2$ with either $k > 2$ or $m > 2$.
6. $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ is residually finite dimensional (RFD).

So far we have

$$(1) \Rightarrow (2)$$

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How do we get back to CEP?

Amenable Traces

Theorem (Connes, Kirchberg)

Let A be a separable unital C^*-algebra. TFAE for a tracial state τ on A.

1. *τ is amenable.*
2. *There exists an embedding $\pi_\tau(A)'' \subset \mathcal{R}^\omega$ such that $\pi_\tau : A \to \pi_\tau(A)'' \subset \mathcal{R}^\omega$ has a ucp lift $A \to \ell^\infty(\mathcal{R})$ and $tr^\omega \pi_\tau = \tau$.*
3. *Given an embedding $A \subset B(\mathcal{H})$, there exists a ucp map $\psi : B(\mathcal{H}) \to \pi_\tau(A)''$ such that $\psi(a) = \pi_\tau(a)$ for all $a \in A$.*

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$$
\begin{tikzcd}
B(\mathcal{H}) \arrow[dr, dashed, "\psi"] & \\
\cup & \\
A \arrow[r, "\pi_\tau"] & \pi_\tau(A)''
\end{tikzcd}
$$

Amenable Traces and WEP

Suppose $A \subset B(\mathcal{H})$ has WEP and tracial state τ.

Amenable Traces and WEP

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$$
\begin{array}{ccc}
B(\mathcal{H}) & & \\
\cup & & \\
A & \xrightarrow{\pi_\tau} & \pi_\tau(A)''
\end{array}
$$

Amenable Traces and WEP

Suppose $A \subset B(\mathcal{H})$ has WEP and tracial state τ.

$$\begin{tikzcd} B(\mathcal{H}) \arrow[r, dashed, "\mathsf{WEP}"] & A^{**} \\ \cup & \\ A \arrow[ur, hook] \arrow[r, "\pi_\tau"] & \pi_\tau(A)'' \end{tikzcd}$$

Amenable Traces and WEP

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$$\begin{tikzcd}
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A \arrow[u, phantom, "\cup"] \arrow[ur, hook] \arrow[r, "\pi_\tau"] & \pi_\tau(A)''
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Corollary

Any trace on a C^*-algebra that has the WEP is amenable.

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Any trace on a C^**-algebra that has the WEP is amenable.*

Hence, if $\mathrm{C}^*(\mathbb{F}_\infty)$ has the WEP, then every trace on $\mathrm{C}^*(\mathbb{F}_\infty)$ is amenable.

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Any trace on a C^*-algebra that has the WEP is amenable.

Hence, if $\mathrm{C}^*(\mathbb{F}_\infty)$ has the WEP, then every trace on $\mathrm{C}^*(\mathbb{F}_\infty)$ is amenable.This will imply CEP.

$\mathrm{C}^*(\mathbb{F}_\infty)$ has WEP $\Rightarrow$ CEP

Proof Outline:

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Suppose M is a II_1-factor with separable predual M_* and faithful tracial state τ.

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Suppose M is a II_1-factor with separable predual M_* and faithful tracial state τ. Since M_* is separable, M contains a countable family of unitaries whose span is dense in the wk*-topology on M.

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If $\mathrm{C}^*(\mathbb{F}_\infty)$ has the WEP, then $\tau\psi$ is amenable, which gives us the embedding

$$\pi_{\tau\psi}(\mathrm{C}^*(\mathbb{F}))'' \subset \mathcal{R}^\omega.$$

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If $\mathrm{C}^*(\mathbb{F}_\infty)$ has the WEP, then $\tau\psi$ is amenable, which gives us the desired embedding

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Consider the following:

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So far we have

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Now, let's talk tensors.

C^*-Tensor Products

Given two unital C^*-algebras A and B, the maximal norm on their algebraic tensor product is given by

$$\|x\|_{\mathsf{max}} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}) \text{ a } *\text{-representation}\}$$

for each $x \in A \odot B$.

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$$A \otimes_{\mathsf{max}} B = \overline{A \odot B}^{\|\cdot\|_{\mathsf{max}}}.$$

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Any $*$-homomorphism $A \odot B \to C$ into a C^*-algebra C extends to a unique $*$-homomorphism $A \otimes_{\mathsf{max}} B \to C$.

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for each $x \in A \odot B$. We write

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Any $*$-homomorphism $A \odot B \to C$ into a C^*-algebra C extends to a unique $*$-homomorphism $A \otimes_{\max} B \to C$. In particular, any pair of $*$-homomorphisms $A \to C$ and $B \to C$ with commuting ranges induces a unique $*$-homorphism $A \otimes_{\max} B \to C$.

C^*-Tensor Products

Given two faithful representations $\pi_1 : A \to B(\mathcal{H}_1)$ and $\pi_2 : B \to B(\mathcal{H}_2)$, we define the spatial norm on $A \odot B$ by

$$\|\sum a_i \otimes b_i\| = \|\sum \pi_1(a_i) \otimes \pi_2(b_i)\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}.$$

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Thanks to Takesaki, we know this is the smallest possible C^*-norm on $A \odot B$, and hence we denote the closure by $A \otimes_{\min} B$.

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Thanks to Takesaki, we know this is the smallest possible C^*-norm on $A \odot B$, and hence we denote the closure by $A \otimes_{\min} B$.

The universal property of $A \otimes_{\max} B$ guarantees a natural surjective $*$-homomorphism $A \otimes_{\max} B \to A \otimes_{\min} B$. When the map is injective, we write

$$A \otimes_{\max} B = A \otimes_{\min} B.$$

Examples of a Unique C^*-tensor norm

Theorem (Choi-Effros, Kirchberg)

A C^*-algebra A is nuclear iff there is a unique C^*-norm on $A \odot B$ for any C^*-algebra B.

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Theorem (Kirchberg)

For any free group $\mathbb{F}$ and any Hilbert space $\mathcal{H}$,

$$\mathrm{C}^*(\mathbb{F}) \otimes_{\mathsf{max}} B(\mathcal{H}) = \mathrm{C}^*(\mathbb{F}) \otimes_{\mathsf{min}} B(\mathcal{H}).$$

Tensor Product Inclusions

If $A \subset B$ and C are C^*-algebras, then there is a natural inclusion

$$C \otimes_{\mathsf{min}} A \subset C \otimes_{\mathsf{min}} B.$$

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However, in general, we cannot expect to have

$$C \otimes_{\text{max}} A \subset C \otimes_{\text{max}} B,$$

i.e. there may be $x \in C \odot A$ for which

$$\|x\|_{C \otimes_{\text{max}} A} > \|x\|_{C \otimes_{\text{max}} B}.$$

Tensor Product Characterization of WEP

Theorem (Lance)

The following are equivalent for a unital C^*-algebra A,

1. *For any C^*-algebras B and C with $A \subset B$, there is a natural inclusion*

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$$\mathrm{C}^*(\mathbb{F}) \otimes_{\max} A \subset \mathrm{C}^*(\mathbb{F}) \otimes_{\max} B.$$
3. *A has the WEP.*

Tensor Product Characterization of WEP

Theorem (Kirchberg)

For any free group $\mathbb{F}$ and any Hilbert space $\mathcal{H}$,

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For any C^*-algebra A and any nonabelian free group $\mathbb{F}$, A has the WEP iff $\mathrm{C}^*(\mathbb{F}) \otimes_{\mathsf{max}} A = \mathrm{C}^*(\mathbb{F}) \otimes_{\mathsf{min}} A$.

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($\Rightarrow$) Embed $A \subset B(\mathcal{H})$. Then

$$\begin{array}{ccc}
\mathrm{C}^*(\mathbb{F}) \otimes_{\max} A & \subset & \mathrm{C}^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}) \\
 & & \| \\
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Corollary

For any free group $\mathbb{F}$, $\mathrm{C}^*(\mathbb{F})$ has the WEP iff

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For any free group $\mathbb{F}$*,* $\mathrm{C}^*(\mathbb{F})$ *has the WEP iff*

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Remark

If any non-abelian free group C^**-algebra has WEP, then all have WEP.*

Scorecard

Consider the following:

1. Connes' Embedding Problem
2. Every C^*-algebra is QWEP.
3. $\mathrm{C}^*(\mathbb{F})$ has WEP for any (every) non-abelian free group $\mathbb{F}$.
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5. $\mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*k}) = \mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*k})$ for each $k, m \geq 2$ with either $k > 2$ or $m > 2$.
6. $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ is residually finite dimensional (RFD).

So far we have

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$$

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So, how do the $\mathbb{Z}_m^{*k}$'s come into the picture?

Now with $\mathbb{Z}_m^{*k}$!

The free group C^*-algebras belong to a larger class of C^*-algebras that "characterize the WEP." We say B characterizes the WEP if for any C^*-algebra A, A has the WEP iff

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Using results of Kirchberg, Boca, and Pisier, one can readily show that $\mathrm{C}^*(\mathbb{Z}_m^{*k})$ characterizes the WEP when either $m > 2$ or $k > 2$.

Now with $\mathbb{Z}_m^{*k}$!

That means, $\mathrm{C}^*(\mathbb{F})$ has WEP iff $\mathrm{C}^*(\mathbb{Z}_m^{*k})$ has WEP iff

$$\mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\mathsf{max}} \mathrm{C}^*(\mathbb{Z}_m^{*k}) = \mathrm{C}^*(\mathbb{Z}_m^{*k}) \otimes_{\mathsf{min}} \mathrm{C}^*(\mathbb{Z}_m^{*k}).$$

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[Scholz, Werner] "The problem of Tsirelson is now to decide the question whether all quantum correlation functions between two independent observers derived from commuting observables can also be expressed using observables defined on a Hilbert space of tensor product form."

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Where does residual finite dimensionality come in?

Finite Dimensional Approximations

A C^*-algebra is *residually finite dimensional* (RFD) if it has a separating family of finite dimensional representations

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Example

1. $\mathrm{C}^*(\mathbb{F}_n)$ is RFD for $2 \leq n \leq \infty$. (Choi '80)
2. $\mathrm{C}^*(\mathbb{Z}_m^{*k})$ is RFD for each $m, k \geq 2$. (Exel-Loring '92)

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Proposition

If A and B are RFD C^*-algebras, then $A \otimes_{\min} B$ is RFD, and $A \otimes_{\max} B$ is RFD iff $A \otimes_{\max} B = A \otimes_{\min} B$.

Finite Dimensional Approximations

So the QWEP conjecture is equivalent to asking whether or not

$$\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2) \simeq \mathrm{C}^*(\mathbb{F}_2) \otimes_{\mathsf{max}} \mathrm{C}^*(\mathbb{F}_2)$$

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Theorem (Scholz, Werner)

[Tsirelson's] problem is equivalent to the question whether all quantum correlation functions can be approximated by correlation function derived from finite-dimensional systems.

Thanks!

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The Lifting Property (again)

Recall that a crucial component of our proof that QWEP $\Rightarrow$ $\mathrm{C}^*(\mathbb{F})$ WEP was that free group C^*-algebras have a so-called lifting property.

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Recall that a crucial component of our proof that QWEP $\Rightarrow \mathrm{C}^*(\mathbb{F})$ WEP was that free group C^*-algebras have a so-called lifting property.

Let A, B, and C be unital C^*-algebras and $\pi : B \twoheadrightarrow C$ a surjective $*$-homomorphism. A ucp map $\phi : A \to C$ is *liftable* if there exists a ucp map $\psi : A \to B$ such that $\pi\psi = \phi$.

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ A \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"'] & C \end{tikzcd}$$

The Lifting Property

A ucp map $\phi : A \to C$ is *locally liftable* if for any finite dimensional operator system $S \subset A$, the restriction $\phi|_S$ has a ucp lift $\psi : S \to B$.

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A unital C^*-algebra has the *(local) lifting property* (L)LP if any ucp map from A into any C^*-quotient is (locally) liftable.

Examples

Corollary (Choi-Effros)

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Let $\mathbb{F}$ *be any free group. Then* $\mathrm{C}^*(\mathbb{F})$ *has the LLP, and it has the LP when* $\mathbb{F}$ *is countable.*

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Corollary

Let A be a unital C^*-algebra and $\mathbb{F}$ a free group such that $\mathrm{C}^*(\mathbb{F})$ surjects onto A. Then A has the LLP iff id_A is locally liftable.

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Back to Tensors

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is guaranteed to be exact, but the sequence

$$0 \to C \otimes_{\mathsf{min}} J \to C \otimes_{\mathsf{min}} B \to C \otimes_{\mathsf{min}} A \to 0$$

may fail to be.

Effros-Haagerup

Theorem (Effros-Haagerup)

Let B be a C^*-algebra and $J \lhd B$. The following are equivalent

1. *For any C^*-algebra C, the sequence*

$$0 \to C \otimes_{\min} J \to C \otimes_{\min} B \to C \otimes_{\min} B/J \to 0$$

   is exact.

2. *The sequence*

$$0 \to B(\mathcal{H}) \otimes_{\min} J \to B(\mathcal{H}) \otimes_{\min} B \to B(\mathcal{H}) \otimes_{\min} B/J \to 0$$

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3. *$\mathrm{id}_{B/J}$ is locally liftable.*

Consider $B = \mathrm{C}^*(\mathbb{F})$ and $\mathrm{C}^*(\mathbb{F})/J = A$.

Tensorial Characterization of the LLP

If $B = \mathrm{C}^*(\mathbb{F})$ and $\mathrm{C}^*(\mathbb{F})/J = A$, then the sequence

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Since $\otimes_{\max}$ is exact and $\mathrm{C}^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}) = \mathrm{C}^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H})$, we know

$$\begin{tikzcd}
B(\mathcal{H}) \otimes_{\min} J \arrow[r, hook] \arrow[d, equal] & B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}) \arrow[r, two heads] \arrow[d, equal] & \frac{B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F})}{B(\mathcal{H}) \otimes_{\min} J} \arrow[d, equal] \\
B(\mathcal{H}) \otimes_{\max} J \arrow[r, hook] & B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}) \arrow[r, two heads] & B(\mathcal{H}) \otimes_{\max} A
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Tensorial Characterization of the LLP

If $B = \mathrm{C}^*(\mathbb{F})$ and $\mathrm{C}^*(\mathbb{F})/J = A$, then the sequence

$$0 \to B(\mathcal{H}) \otimes_{\min} J \to B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}) \to B(\mathcal{H}) \otimes_{\min} A \to 0$$

is exact iff A has the LLP.

Since $\otimes_{\max}$ is exact and $\mathrm{C}^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}) = \mathrm{C}^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H})$, we know

$$\begin{tikzcd}
B(\mathcal{H}) \otimes_{\min} J \arrow[r, hook] \arrow[d, equal] & B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}) \arrow[r, two heads] \arrow[d, equal] & \frac{B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F})}{B(\mathcal{H}) \otimes_{\min} J} \arrow[d, equal] \\
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\end{tikzcd}$$

It follows that A has LLP iff $B(\mathcal{H}) \otimes_{\max} A = B(\mathcal{H}) \otimes_{\min} A$.

Tensorial Duality of WEP and LLP

Theorem (Kirchberg)

For any C^*-algebras A and B, we have the following for any infinite dimensional Hilbert space $\mathcal{H}$ and any nonabelian free group $\mathbb{F}$

1. *A has the LLP iff $B(\mathcal{H}) \otimes_{\max} A = B(\mathcal{H}) \otimes_{\min} A$.*
2. *B has the WEP iff $B \otimes_{\max} \mathrm{C}^*(\mathbb{F}) = B \otimes_{\min} \mathrm{C}^*(\mathbb{F})$.*
3. *If A has the LLP and B has the WEP, then $A \otimes_{\max} B = A \otimes_{\min} B$.*

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For (3), embed $B \subset B(\mathcal{H})$. Then we have

$$\begin{tikzcd} A \otimes_{\max} B(\mathcal{H}) \arrow[r, equal, "LLP"] & A \otimes_{\min} B(\mathcal{H}) \\ A \otimes_{\max} B \arrow[u, hook, "WEP"] & A \otimes_{\min} B \arrow[u, hook] \end{tikzcd}$$

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Characterizing the WEP

We say a C^*-algebra B characterizes the WEP if for any C^*-algebra A, A has the WEP iff

$$A \otimes_{\max} B = A \otimes_{\min} B.$$

Using Kirchberg's Duality theorem, one can readily show that B characterizes the WEP if B has the LLP and $\mathrm{C}^*(\mathbb{F}_2)$ embeds into B as the image of a conditional expectation.

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Indeed, let A be any C^*-algebra, if A has WEP, then by Kirchberg's duality theorem, $B \otimes_{\mathsf{max}} A = B \otimes_{\mathsf{min}} A$. Conversely, since

$$\mathrm{C}^*(\mathbb{F}_2) \otimes_{\mathsf{max}} A \subset B \otimes_{\mathsf{max}} A,$$

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See [6] and [9] for some examples.

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Proposition

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When either $m > 2$ or $k > 2$, we can find a copy of $\mathbb{F}_2$ inside $\mathbb{Z}_m^{*k}$. Consequently, $\mathrm{C}^*(\mathbb{F}_2)$ embeds into $\mathrm{C}^*(\mathbb{Z}_m^{*k})$ as the range of a conditional expectation.