The abstract approach to classifying C^*-algebras
Part I: Classifying algebras

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WWU Münster

Actions of Tensor Categories on C^*-algebras
IPAM, January 21-28, 2021

Goal/Caveat

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Theorem (2015, Many hands)

Simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by K-theory and traces.

A tale of two algebras

C^*- and W^*-algebras

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In the commutative case,

C^*-algebras
are $C_0(X)$,
locally compact Hausdorff X.

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Even in the noncommutative setting,

C^*-algebras
are more topological.

W^*-algebras
are more measure-theoretic.

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- $\mathrm{L}(\mathbb{Z}) = L^\infty(\mathbb{T}) \simeq L^\infty(\mathbb{T}^2) = \mathrm{L}(\mathbb{Z}^2)$

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A similar construction with a regular Borel probability space (Y, μ) with probability measure preserving (pmp) $\beta : \Gamma \curvearrowright Y$ yields $L^\infty(Y) \rtimes_\beta \Gamma$.

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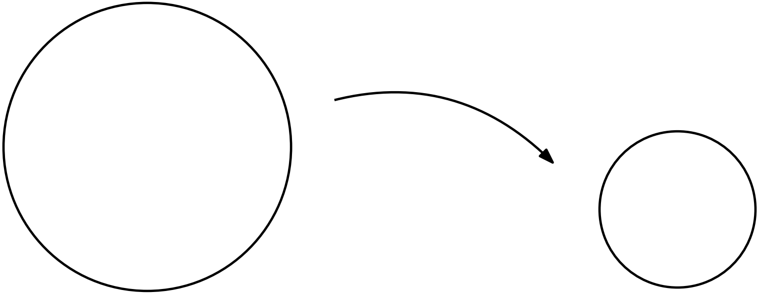
Example

$\mathbb{Z} \curvearrowright \mathbb{T}$ by irrational rotation, $\theta : \mathrm{e}^{2\pi \mathrm{i} t} \mapsto \mathrm{e}^{2\pi \mathrm{i}(\theta + t)}$.
$\rightsquigarrow$ *Irrational rotation algebras* $A_\theta := C(\mathbb{T}) \rtimes_{\lambda,\theta} \mathbb{Z}$ and $L^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.

Classification



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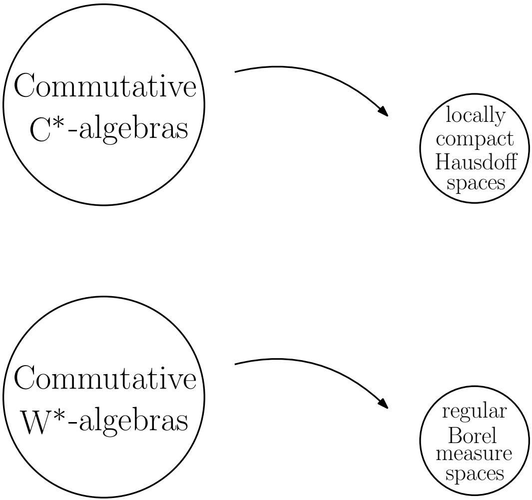
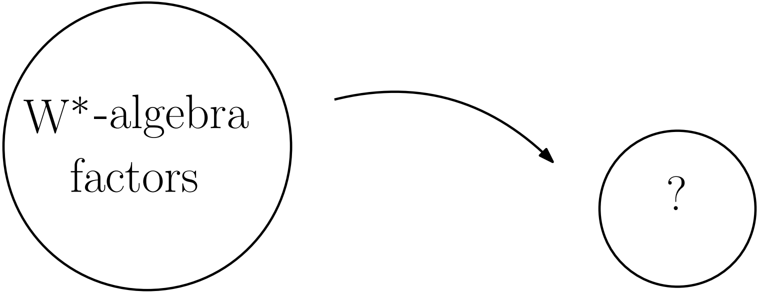


W^*-Classification

W*-Classification: Factors



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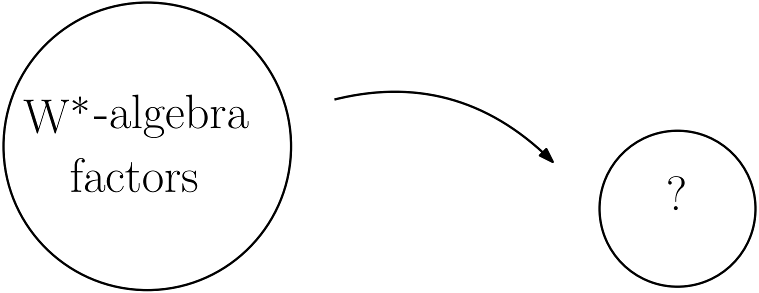

Let's focus on the story for II_1-factors.

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- $\mathrm{L}(\Gamma)$ for an infinite conjugacy class (ICC) group Γ

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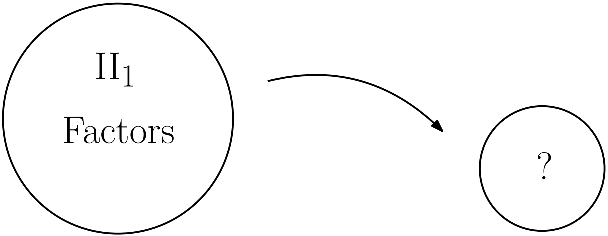
$$\begin{array}{ccccccc}
\mathrm{M}_2 & \hookrightarrow & \mathrm{M}_2 \otimes \mathrm{M}_2 & \hookrightarrow & \ldots & \hookrightarrow & \overline{\bigotimes_{k=1}^{\infty} \mathrm{M}_2}^{SOT} \\
\updownarrow & & \updownarrow & & & & \updownarrow \\
\mathrm{M}_2 & \xhookrightarrow{a \mapsto a \oplus a} & \mathrm{M}_4 & \hookrightarrow & \ldots & \hookrightarrow & \overline{\bigcup_{k=1}^{\infty} \mathrm{M}_{2^k}}^{SOT}
\end{array}$$

The closure is with respect to the GNS representation for $\tau := \otimes_{k=1}^{\infty} \tau_2$, with τ_2 the normalized trace on M_2.

W*-Classification: II_1-factors



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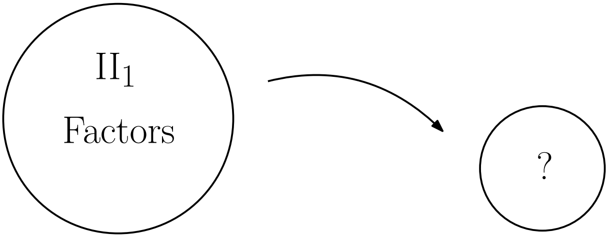

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For instance, we still don't know if $\mathrm{L}(\mathsf{F}_2) \simeq \mathrm{L}(\mathsf{F}_3)$.

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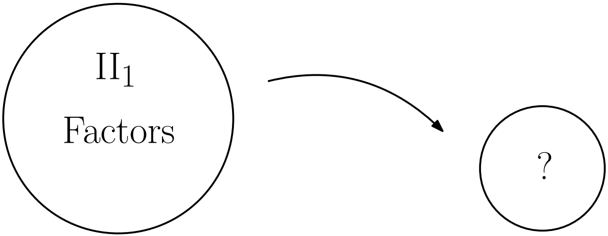

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We need an additional "smallness criteria".

Smallness Criteria 1: Approximately Finite

A separably acting von Neumann algebra is *Approximately Finite Dimensional* (AFD) if it is the SOT closure of an increasing union of finite dimensional subalgebras.

A tracial AFD von Neumann algebras is called *hyperfinite*.

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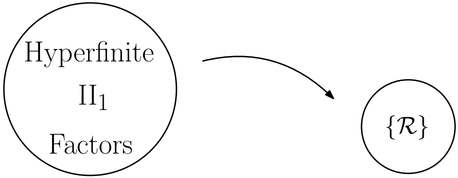
We denote it by $\mathcal{R}$.



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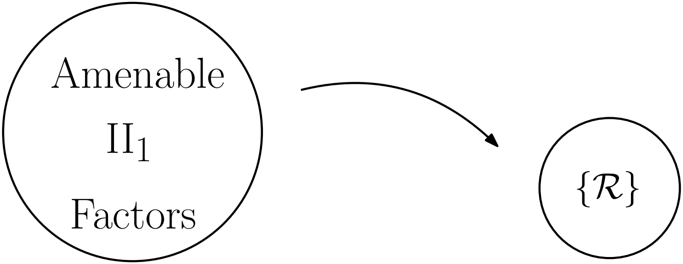
With Murray and von Neumann's classification of hyperfinite II_1-factors:



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An important factor in Connes work is the fact that any (separably acting) amenable II_1-factor $\mathcal{M}$ is *McDuff*, i.e.,

$$\mathcal{M} \simeq \mathcal{M} \bar{\otimes} \mathcal{R}.$$

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We consider simple, separable, infinite dimensional C^*-algebras (with tracial states), but how do the "smallness criteria" translate to this setting?

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Caution: $L^\infty([0,1])$ is hyperfinite, but $C([0,1])$ is not AF.

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Operator algebraic K-theory: noncommutative extension of topological K-theory of Atiyah and Hirzebruch.

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$K_0(A)$ is an ordered abelian group which captures the structure of projections in a C^*-algebra A and its matrix amplifications $\mathsf{M}_n(A)$.

For A unital, $K_0(A)$ is the Grothendieck group of its MvN semigroup.

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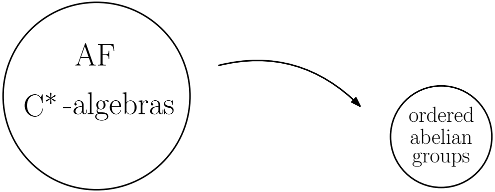

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A satisfies the *completely positive approximation property* (CPAP) if there exist completely positive contractive (cpc) maps $A \xrightarrow{\psi_i} \mathsf{M}_{n_i} \xrightarrow{\varphi_i} A$ so that

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A map $\phi : A \to B$ is *completely positive* if $\phi^{(n)}(\mathsf{M}_n(A)_+) \subset \mathsf{M}_n(B)_+ \ \forall\ n$.

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$K_1(A)$ is an abelian group which captures the structure of unitaries in a C^*-algebra A and its matrix amplifications $\mathsf{M}_n(A)$.

For A unital, $K_1(A) = \mathcal{U}_\infty(A)/\sim_h$ where $\mathcal{U}_\infty(A) = \bigcup_n \mathcal{U}_n(A)$.

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  Bott Periodicity $\rightsquigarrow$ higher K-groups are redundant.

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- The final ingredient in the invariant is traces, i.e. the simplex of tracial states $T(A)$ of a C^*-algebra.

Together, these are referred to as *"K-theory and traces"*.

Classification by K-Theory and Traces?

What is the class of *classifiable* C^*-algebras, i.e., those that can be classified by K-theory and traces?

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$\rightsquigarrow$ We have to disregard commutative and reduced group C^*-algebras.

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  There are infinitely many simple, separable, unital, exact, non-nuclear C^*-algebras that are indistinguishable from $\mathcal{O}_2$ using just K-theory and traces.

But we keep simple AF, irrational rotation algebras, Cuntz algebras, and crossed products with a free minimal action.

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In particular, Rørdam gives and example of a simple, separable, unital, nuclear C^*-algebra that is finite but has no traces.

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We need additional structural criteria.

Finite nuclear dimension:
Or how I learned to stop worrying and love $\mathcal{Z}$

Classification: Finite nuclear dimension

Theorem (2015, Many hands)

Simple, separable, unital, infinite dimensional, nuclear C^**-algebras with finite nuclear dimension in the UCT class are classified by K-theory and traces.*

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Theorem (Castillejos, Evington, Tikuisis, White, Winter^2)

A simple, separable, unital, infinite dimensional, nuclear C^*-algebra A has finite nuclear dimension iff it is stable with respect to tensoring with the Jiang-Su algebra $\mathcal{Z}$, i.e.

$$A \otimes \mathcal{Z} \simeq A.$$

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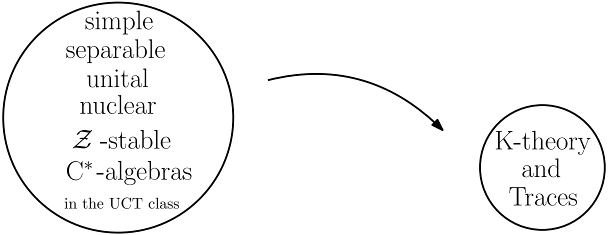

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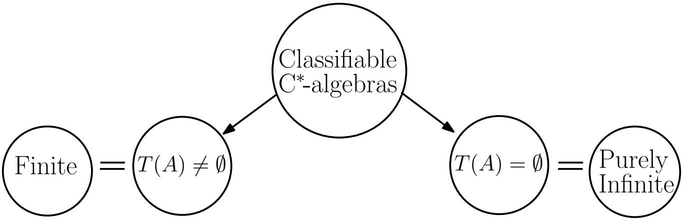

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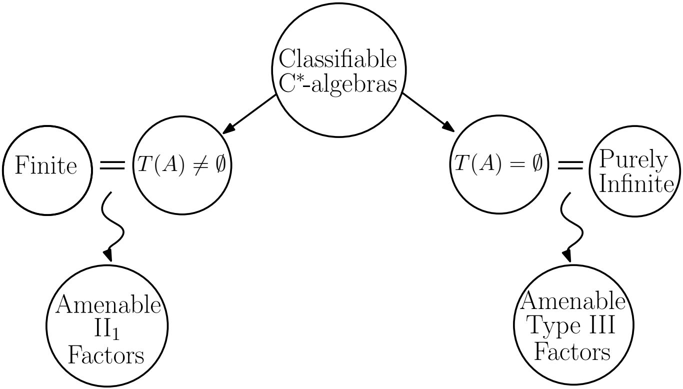

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  $\mathcal{Z}$ also satisfies the UCT, so it is classifiable.

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Any unital C^*-algebra A has the same K-theory and traces as $A \otimes \mathcal{Z}$.

$\rightsquigarrow$ We can only classify *up to* $\mathcal{Z}$-stability, i.e., $\mathcal{Z}$-stability is necessary for classification.

But what **is** $\mathcal{Z}$??

$\mathcal{Z}$ can be constructed as the inductive limit of certain so-called *dimension drop* algebras:

$$Z_{p,q} = \{f \in C([0,1], \mathrm{M}_p \otimes \mathrm{M}_q) : f(0) \in \mathrm{M}_p, f(1) \in \mathrm{M}_q\},$$

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Actually, we are mostly interested in $\mathcal{Z}$-stability.

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*-homomorphisms $\psi, \phi : A \to B$ are a.u.e. if $\exists\ (u_n)_n \subset \mathcal{U}(B)$ such that

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$\rightsquigarrow$ The class of simple, separable, unital, nuclear C^*-algebras localized at $\mathcal{Z}$ yields the class of classifiable C^*-algebras (modulo UCT).

About the UCT

We also require that a classifiable C^*-algebra satisfies the Universal Coefficient Theorem (UCT).

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Kasparov's $KK(\cdot, \cdot)$ is a bivariant functor on separable C^*-algebras, generalizing both K-homology and K-theory.

Think of KK-equivalence as a loose notion of homotopy equivalence.

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The UCT effectively allows us to state the invariant in terms of K-theory but then utilize KK-theory for the proofs.

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The UCT Problem:

- **Open Question:** Does every separable, nuclear C^*-algebra satisfy the UCT? What about every classifiable C^*-algebra?
- The answer is yes for virtually any example one can write down.
- (Barlak-Li, Tu) A separable, nuclear C^*-algebra satisfies the UCT if it has a Cartan subalgebra.

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- $C(X)$ o$_{\lambda,\alpha}$ Γ arising from free minimal actions of groups with local subexponential growth on finite-dimensional spaces (Many Hands $\neq$ the hands in the classification)

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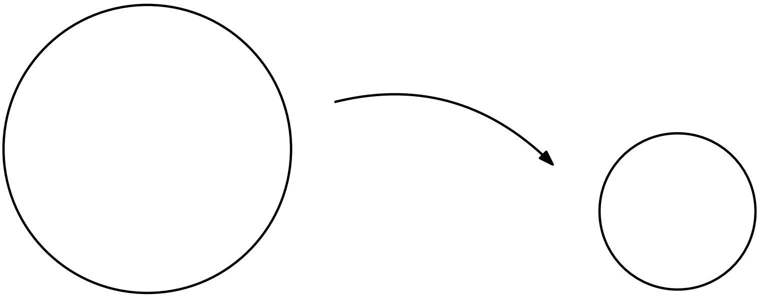
- Simple, unital, infinite dimensional AF algebras,
- Irrational rotation algebras A_θ,
- Cuntz algebras $\mathcal{O}_n$,
- $\mathcal{Z}$-stabilization of Rørdam's and Tom's counterexamples.
- The C^*-algebra $\mathrm{C}^*_\pi(\Gamma)$ generated by an irreducible unitary representation π of a finitely generated nilpotent group, (Eckhardt-Gillaspy)
- $C(X) \rtimes_{\lambda,\alpha} \Gamma$ arising from free minimal actions of groups with local subexponential growth on finite-dimensional spaces (Many Hands $\neq$ the hands in the classification)

"I am prepared to stick my neck out and say that this should hold for all amenable groups - though that's still a long way off."– S. White.

Thanks!



Appendix

Operator K-theory: K_0

Operator algebraic K-theory is the noncommutative extension of topological K-theory of Atiyah and Hirzebruch.

Suppose A is a unital C^*-algebra.

$K_0(A)$ is an ordered abelian group which captures the structure of projections in A and its matrix amplifications $\mathsf{M}_n(A)$.

More precisely, it is the Grothendieck group of its MvN semigroup of projections:

$$\{p \in \textstyle\bigcup_n \mathsf{M}_n(A) : p \text{ a projection }\} \,/\, \sim_{\mathsf{MvN}} .$$

Operator K-theory: K_1

Suppose A is a unital C^*-algebra.

$K_1(A)$ is an abelian group which captures the structure of unitaries in C^*-algebra A and its matrix amplifications $\mathsf{M}_n(A)$.

More precisely, writing $\mathcal{U}_\infty(A) := \bigcup_n \mathcal{U}_n(A)$,

$$K_1(A) := \mathcal{U}_\infty(A)/ \sim_h .$$

For $n \geq 0$, we have $K_n(A) \simeq K_{n+2}(SA)$ where $SA := C_0(0,1) \otimes A$.

Bott Periodicity: $K_1(A) \simeq K_0(SA)$ and $K_0(A) \simeq K_1(SA)$.
$\rightsquigarrow$ higher K-groups are redundant.

KK-Theory

Kasparov's $KK(\cdot,\cdot)$ is a bivariant functor on separable C^*-algebras, generalizing both K-homology and K-theory.

Think of KK-equivalence as a loose notion of homotopy equivalence.

(Cuntz) $KK(A,B)$ is an abelian group consisting of homotopy classes of pairs of $*$-homomorphisms $A \to M(B \otimes \mathcal{K})$ who agree modulo $B \otimes \mathcal{K}$.

$\rightsquigarrow$ We can consider a category whose objects are separable C^*-algebras and whose morphisms are KK-elements, i.e. Cuntz pairs. Here $KK(A,B)$ are Hom sets, and isomorphisms are *KK-equivalences*.

Rørdam's Examples

Rørdam constructs

- A simple, separable, unital, nuclear, UCT-class that contains a (nonzero) finite projection and an infinite projection.
- B simple, separable, unital, nuclear, UCT-class such that B is finite but $M_2(B)$ is (properly) infinite (in particular B is finite but not stably finite).
- Moreover, B is finite but has no traces.