TENSOR LECTURE 1

This is sourced heavily from Sections [2, 3.1,3.2,3.3,3.5] and the GOALS Lecture notes.

1. Algebraic Tensor products

Tensor products are an important construction in operator algebras. Generally, one should think of the tensor product of two vector spaces as a sort of product of the spaces themselves. However, unlike direct products, tensor products allow for more interaction between elements in the spaces.

Tensor products are already an important construction for vector spaces and algebras. So, we begin by extracting a few important facts there before we add on any topological information. A fairly standard practice is to use different notation for algebraic tensor products and tensor products with some extra topological information. We adopt the convention of writing $A \odot B$ for the algebraic tensor product of two (potentially Banach) algebras (i.e., just their tensor product as plain 'ol algebras) and $A \otimes B$ when the algebraic tensor product is completed with respect to some topology.

1.1 Tensor products of $\mathbb{C}$-$*$-algebras We begin by considering C*-algebras as just complex *-algebras. While tensor products are usually defined via a universal property with respect to bilinear maps on direct products,

Definition 1.1. *Let A and B be $\mathbb{C}$-vector spaces. Their* **tensor product** *is a vector space $A \odot B$, together with a bilinear map $\odot : A \times B \to A \odot B$, such that $A \odot B$ is universal in the following sense:*
For any $\mathbb{C}$-vector space C and any bilinear map $\phi : A \times B \to C$, there exists a unique bilinear map $\tilde{\phi} : A \odot B \to C$ so that $\tilde{\phi}(a \odot b) = \phi(a, b)$ for all $a \in A$ and $b \in B$.

we usually think of them in the following way:

Definition 1.2. *Given $*$-algebras A and B, their (algebraic)* tensor product *$A \odot B$ is the *-algebra consisting of formal linear combinations of elements of the form $a \odot b$ for $a \in A$ and $b \in B$ such that the following relations are satisfied for all $a_1, a_2, a \in A$, $b_1, b_2, b \in B$, and $\lambda \in \mathbb{C}$*

$$\begin{aligned} (a_1 + a_2) \odot b &= (a_1 \odot b) + (a_2 \odot b), \\ a \odot (b_1 + b_2) &= (a \odot b_1) + (a \odot b_2), \quad \textit{and} \\ \lambda(a \odot b) &= (\lambda a) \odot b = a \odot (\lambda b), \end{aligned}$$

where multiplication and adjoints are defined by

$$\begin{aligned} (a_1 \odot b_1)(a_2 \odot b_2) &= (a_1 a_2) \odot (b_1 b_2) \quad \textit{and} \\ (a \odot b)^* &= a^* \odot b^* \end{aligned}$$

and extendend linearly.

Elements of the form $a \odot b$ for $a \in A$ and $b \in B$ are called *simple tensors*. That is, $A \odot B$ is spanned by its simple tensors.

Remark 1.3. *Although $A \odot B$ is spanned by its simple tensors, it consists of many more elements. For example, in general the element $(a_1 \odot b_1) + (a_2 \odot b_2)$ cannot be written as a simple tensor $a \odot b$. Note also that if $a = 0$ or $b = 0$, then $a \odot b = 0$.*

Proposition 1.4. *If $\{e_i\}_{i \in I}$ is a basis for A and $\{e'_j\}_{j \in J}$ is a basis for B, then $\{e_i \odot e'_j\}_{(i,j) \in I \times J}$ is a basis for $A \odot B$.*

Proposition 1.5. *If $\{e_i\}_{i \in I}$ is a basis for B and $x \in A \odot B$, then there exists a unique finite set $I_0 \subset I$ and $\{a_i\}_{i \in I_0} \subset A$ so that $x = \sum_{i \in I_0} a_i \odot e_i$.*

Example 1.6. *Let A be a $\mathbb{C}$-vector space and fix $m, n \in \mathbb{N}$. Recall that $\{E_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_n(\mathbb{C})$, where $E_{i,j}$ is the matrix with a 1 in the (i,j) entry and zeros elsewhere. The previous proposition therefore implies that every element of $A \odot M_n(\mathbb{C})$ can be written as*

$$\sum_{1 \le i,j \le n} a_{i,j} \odot E_{i,j}.$$

It is helpful to think of the elementary tensor $a_{i,j} \odot E_{i,j}$ as an $n \times n$ matrix with the vector $a_{i,j}$ in the (i,j) entry and zeros elsewhere. From this perspective, the above element becomes

$$\sum_{1 \le i,j \le n} a_{i,j} \odot E_{i,j} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}.$$

That is, we can think of $A \odot M_n(\mathbb{C})$ as $M_n(A)$: the $n \times n$ matrices with entries in A:

$$\mathrm{M}_n(A) := \{[a_{ij}]_{1 \le i,j \le n} \ : \ a_{ij} \in A, 1 \le i,j \le n\} \tag{1.1}$$

The vector space operations on $M_n(A)$ are then determined by the entrywise operations from A. When A is a ($*$-)algebra, this also comes with a natural multiplication (and involution where $[a_{i,j}]^* = [a^*_{j,i}]$ for all $[a_{i,j}] \in \mathrm{M}_n(A)$). We will usually suppress the usual subscripts on the matrices, i.e. we write $[a_{ij}]$ for $[a_{ij}]_{1 \le i,j \le n}$ (sometimes also $[a_{ij}]_{ij}$).

Exercise 1. *Let A be any C^*-algebra, $1 \le n < \infty$, and let $E_{i,j}$ denote the matrix units on $\mathrm{M}_n(\mathbb{C})$ (i.e. the matrices with 1 in the i,j coordinate and 0 elsewhere). Define a map $\pi : \mathrm{M}_n(A) \to \mathrm{M}_n(\mathbb{C}) \odot A$ by $\pi([a_{i,j}]) = \sum_{i,j=1}^n E_{i,j} \odot a_{ij}$. Show that this is an algebraic $*$-isomorphism.*

Example 1.7. *For $m, n \ge 1$, we have $M_n(\mathbb{C}) \odot M_m(\mathbb{C}) \cong M_{nm}(\mathbb{C})$. For $A = [a_{ij}] \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$, the matrix array for $A \odot B \in B(\mathbb{C}^2 \otimes \mathbb{C}^3)$, called the* Kronecker product *is*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix}$$

Exercise 2. *Verify the following identifications for *-algebras A and B with B unital.*

$$A \simeq A \odot \mathbb{C} \simeq A \odot \mathbb{C}1_B \subset A \odot B.$$

Just as we take tensor products of linear spaces, we can take tensor products of linear maps.[1] The following is more of a proposition/ definition; existence and uniqueness of these maps come from the above universal property.

Proposition 1.8. *Suppose $A_1 A_2, B_1, B_2, B$ are $*$-algebras and $\phi_i : A_i \to B_i$, $i = 1,2$ and $\psi_i : A_i \to B$, $i = 1,2$ are linear maps.*

(1) There is a unique linear map

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2,$$

called the tensor product *of ϕ_1 and ϕ_2, so that $\phi_1 \odot \phi_2(a \odot b) = \phi_1(a) \odot \phi_2(b)$ for all $a \in A_1$, $b \in A_2$.*

(2) There exists a unique linear map

$$\psi_1 \times \psi_2 : A_1 \odot A_2 \to B,$$

called the product *of ψ_1 and ψ_2, so that $\psi_1 \times \psi_2(a \odot b) = \psi_1(a)\psi_2(b)$ for all $a \in A_1$, $b \in A_2$.*

Moreover:

(1) If $\phi_i : A_i \to B_i$, $i = 1,2$ are *-homomorphisms, then $\phi_1 \odot \phi_2$ is a *-homomorphism; and
(2) If $\psi_i : A_i \to B$, $i = 1,2$ are *-homomorphisms, $\psi_1 \times \psi_2$ is a $*$-homomorphism provided that the ranges $\psi_1(A_1)$ and $\psi_2(A_2)$ commute in B, i.e. for each $a_1 \in A_1$ and $a_2 \in A_2$, $\psi_1(a_1)\psi_2(a_2) = \psi_2(a_2)\psi_1(a_1)$.

[1] For those categorically inclined, tensors play well with linear categories and act like "multiplication" for objects/ morphisms.

Example 1.9. *Let $\phi\colon A \to B$ be a linear map between $\mathbb{C}$-vector spaces and fix $m, n \in \mathbb{N}$. If we let I_n denote the identity map on $M_n(\mathbb{C})$, then $\phi \odot I_n\colon A \odot M_n(\mathbb{C}) \to B \odot M_n(\mathbb{C})$. In particular, we have*

$$\phi \otimes I_n \left(\sum_{1 \leq i,j \leq n} a_{i,j} \odot E_{i,j} \right) = \sum_{1 \leq i,j \leq n} \phi(a_{i,j}) \odot E_{i,j}.$$

If we identify $A \odot M_n(\mathbb{C}) \cong M_n(A)$ and $B \odot M_n(\mathbb{C}) \cong M_n(B)$ as in Example 1.6, then

$$\phi \odot I_n \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} \phi(a_{1,1}) & \cdots & \phi(a_{1,n}) \\ \vdots & \ddots & \vdots \\ \phi(a_{n,1}) & \cdots & \phi(a_{n,n}) \end{pmatrix}.$$

That is, $\phi \odot I_n$ is simply the map that applies ϕ to each entry. This map is called a matrix amplification *of ϕ and is denoted by $\phi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B)$.*

The tensor product of linear maps preserves both injectivity and exact sequences:

Proposition 1.10. *Suppose A_1, A_2, B_1, B_2 are $\mathbb{C}$-vector spaces and $\phi_i : A_i \to B_i$, $i = 1, 2$ are linear maps. If ϕ_i and ϕ_2 are injective. Then $\phi_1 \odot \phi_2$ is also injective.*

Proposition 1.11. *Suppose J, A, B, C are $\mathbb{C}$-vector spaces. If $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$ is a short exact sequence (i.e. ι is injective, π is surjective, and $\ker(\pi) = \iota(J)$), then so is*

$$0 \to J \odot C \xrightarrow{\iota \odot id_C} A \odot C \xrightarrow{\pi \odot id_C} B \odot C \to 0.$$

1.2 Tensor products of Hilbert spaces If $\mathcal{H}_1$ and $\mathcal{H}_2$ are a pair of Hilbert spaces, the **tensor product** of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$, is defined as follows. Consider first the algebraic tensor product

$$\mathcal{H}_1 \odot \mathcal{H}_2 = \left\{ \sum_{j=1}^{n} \xi_j \odot \eta_j : n \in \mathbb{N},\ \xi_j \in \mathcal{H}_1, \eta_j \in \mathcal{H}_2 \right\}$$

with inner product

$$\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle := \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle.$$

$\mathcal{H}_1 \otimes \mathcal{H}_2$ is then defined as the completion of $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to metric induced by this inner product.

If we want to emphasize that a simple tensor $\xi \odot \eta$ lives in the completion $\mathcal{H}_1 \otimes \mathcal{H}_2$, we will sometimes write $\xi \otimes \eta$ instead of $\xi \odot \eta$.

Exercise 3. *Show that the norm in $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfies $\|\xi \odot \eta\| = \|\xi\| \|\eta\|$.*

Exercise 4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with orthonormal bases $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$. Then $\{e_i \odot f_j\}_{(i,j) \in I \times J}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Use this to show that If $\{e_i\}_{i \in I}$ is a basis for A and $\{e'_j\}_{j \in J}$ is a basis for B, a basis for $A \odot B$.*

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \cong \ell^2(J, \mathcal{H}_1) \cong \ell^2(I, \mathcal{H}_2).$$

Exercise 5. *Let $\mathcal{H}$ be a Hilbert space. Show that $\ell^2(I) \otimes \mathcal{H} \cong \ell^2(I, \mathcal{H})$.*

Exercise 6. *Let $\mathcal{H}$ be a Hilbert space and fix $n \in \mathbb{N}$. Show that $\mathcal{H} \otimes \mathbb{C}^n \cong \mathcal{H}^n$.*

1.3 Tensor Products of Bounded Operators on Hilbert Spaces Given operators $x_i \in B(\mathcal{H}_i)$ for $i = 1, 2$, we have a natural algebraic tensor product mapping $x_1 \odot x_2 : \mathcal{H}_1 \odot \mathcal{H}_2 \to \mathcal{H}_1 \odot \mathcal{H}_2$ given on simple tensors by

$$(x_1 \odot x_2)(\xi \odot \eta) = x_1 \xi \odot x_2 \eta.$$

This extends linearly to a linear map $\mathcal{H}_1 \odot \mathcal{H}_2 \to \mathcal{H}_1 \odot \mathcal{H}_2$ which is defined on sums of simple tensors by

$$x_1 \odot x_2 \left(\sum_{1}^{n} c_j (\xi_j \odot \eta_j) \right) = \sum_{1}^{n} c_j (x_1 \xi_j \odot x_2 \eta_j).$$

The map $x_1 \odot x_2$ extends to an operator in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by the following proposition. To emphasise that this extension is defined on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we denote it as $x_1 \otimes x_2$.

Proposition 1.12. *Given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and operators $x_i \in B(\mathcal{H}_i)$, $i = 1, 2$, there is a unique linear operator $x_1 \otimes x_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that*

$$x_1 \otimes x_2(\xi_1 \odot \xi_2) = x_1\xi_1 \odot x_2\xi_2$$

for all $\xi_i \in \mathcal{H}_i$, $i = 1, 2$, and moreover $\|x_1 \otimes x_2\| = \|x_1\|\|x_2\|$.

Proof. First, we want to show that the operator $x_1 \odot x_2$ is bounded on $\mathcal{H}_1 \odot \mathcal{H}_2$, which means we can extend it to a bounded operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Assume for now that $x_2 = 1_{\mathcal{H}_2}$, and write $x = x_1$. Let $\sum_1^n c_j(\xi_j \odot \eta_j) \in \mathcal{H}_1 \odot \mathcal{H}_2$. Using a Gram-Schmidt process, we may assume η_j are orthonormal (check). Then we compute

$$\begin{aligned}
\left\|x \odot 1_{\mathcal{H}_2}(\sum_{j=1}^n c_j(\xi_j \odot \eta_j))\right\|^2 &= \left\|\sum_{j=1}^n c_j x\xi_j \odot \eta_j\right\|^2 = \left|\langle \sum_{i=1}^n c_i x\xi_i \odot \eta_i, \sum_{j=1}^n c_j x\xi_j \odot \eta_j\rangle\right| \\
&= \left|\sum_{i=1}^n \sum_{j=1}^n c_i\bar{c}_j \langle x\xi_i, x\xi_j\rangle\langle \eta_i, \eta_j\rangle\right| = \sum_{j=1}^n |c_j|^2 \|x\xi_j\|^2 \le \|x\|^2 \sum_{j=1}^n |c_j|^2\|\xi_j\|^2 \\
&= \|x\|^2 \left|\sum_{i=1}^n \sum_{j=1}^n c_i\bar{c}_j \langle \xi_i, \xi_j\rangle\langle \eta_i, \eta_j\rangle\right| = \|T\|^2 \left\|\sum_{j=1}^n c_j(\xi_j \odot \eta_j)\right\|^2 .
\end{aligned}$$

Then $\|x \odot 1_{\mathcal{H}_2}\| \le \|x\|$ on $\mathcal{H}_1 \odot \mathcal{H}_2$, meaning it extends to an operator in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, denoted by $x \otimes 1_{\mathcal{H}_2}$, with $\|x \otimes 1_{\mathcal{H}_2}\| \le \|x\|$. Similarly, one shows that for any $x_2 \in B(\mathcal{H}_2)$, we have $1_{\mathcal{H}_1} \otimes x_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Now, for $x_1 \in B(\mathcal{H}_1)$ and $x_2 \in B(\mathcal{H}_2)$, we compose $(1_{\mathcal{H}_1} \otimes x_2)(x_1 \otimes 1_{\mathcal{H}_2})$ to get $x_1 \otimes x_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\|x_1 \otimes x_2\| \le \|x_1\|\|x_2\|$ and

$$x_1 \otimes x_2(\xi_1 \otimes \xi_2) = x_1\xi_2 \otimes x_2\xi_2$$

for all $\xi_1 \otimes \xi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$. To show that, in fact, we have $\|x_1 \otimes x_2\| = \|x_1\|\|x_2\|$, we find, for any $\varepsilon > 0$, unit vectors $\xi_i \in \mathcal{H}_i$ with $\|x_i\xi_i\| \ge \|x_i\| + \epsilon$ for $i = 1, 2$. Then, using Exercise 3, we have

$$\|x_1 \otimes x_2\| \ge \|(x_1 \otimes x_2)(\xi_1 \otimes \xi_2)\| = \|x_1\xi_1 \otimes x_2\xi_2\| = \|x_1\xi_1\|\|x_2\xi_2\| \ge (\|x_1\| + \epsilon)(\|x_2\| + \epsilon).$$

Letting $\epsilon \to 0$ yields the claimed equality. □

We will take for granted that taking tensor products of operators is well-behaved with respect to addition, (scalar) multiplication, and adjoints.

In infinite dimensions, we do not have $B(\mathcal{H}_1) \odot B(\mathcal{H}_2) = B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (the former is no longer automatically closed). What we can say is that $B(\mathcal{H}_1) \cong B(\mathcal{H}_1) \otimes \mathbb{C}1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \cong \mathbb{C}1_{\mathcal{H}_1} \otimes B(\mathcal{H}_2)$, and Proposition 1.12 gives a natural *-homomorphism

$$B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

Proposition 1.13. *For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we define $*$-homomorphisms $\iota_i : B(\mathcal{H}_i) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by identifying $B(\mathcal{H}_1) \simeq B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \simeq \mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$. These induce a product $*$-homomorphism $\iota_1 \times \iota_2 : B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, which is injective.*

Proof. Since $B(\mathcal{H}_1) \simeq B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $B(\mathcal{H}_2) \simeq \mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$ (**Exercise:** check) and $B(\mathcal{H}_1) \odot \mathbb{C}1_{\mathcal{H}_2}$ and $\mathbb{C}1_{\mathcal{H}_1} \odot B(\mathcal{H}_2)$ commute in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (**Exercise:** check), we have from Section 1.2 the product $*$-homomorphism

$$B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2),$$

given by

$$\sum_{j=1}^n x_j \odot y_j \mapsto \sum_{j=1}^n (x_j \otimes 1_{\mathcal{H}_2})(1_{\mathcal{H}_1} \otimes y_i) = \sum_{j=1}^n x_j \otimes y_j.$$

We just need to show that this map is injective, i.e. if the operator $\sum_{j=1}^n x_j \otimes y_j \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is zero, then the sum of elementary tensors $\sum_{j=1}^n x_j \odot y_j \in B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ is also zero. By possibly re-writing the coefficients of the x_j, we may assume that the operators $\{x_j\}$ are linearly independent. If $0 = \sum_{j=1}^n x_j \otimes y_j \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, then for all vectors $\xi_1, \eta_1 \in \mathcal{H}_1$ and $\xi_2, \eta_2 \in \mathcal{H}_2$, we have

$$\begin{aligned} 0 &= \langle (\sum_{j=1}^n x_j \otimes y_j)(\xi_1 \otimes \xi_2), (\eta_1 \otimes \eta_2) \rangle = \sum_{j=1}^n \langle x_j \xi_1 \otimes y_j \xi_2, \eta_1 \otimes \eta_2 \rangle \\ &= \sum_{j=1}^n \langle x_j \xi_1, \eta_1 \rangle \langle y_j \xi_2, \eta_2 \rangle = \sum_{j=1}^n \langle \big(\langle y_j \xi_2, \eta_2 \rangle \big) x_j \xi_1, \eta_1 \rangle. \end{aligned}$$

Since this holds for all $\xi_1, \eta_1 \in \mathcal{H}_1$ the operator $\sum_{j=1}^n \langle y_j \xi_2, \eta_2 \rangle x_j \in B(\mathcal{H}_1)$ is zero. Since we assumed the $\{x_j\}$ are linearly independent, the coefficients $\langle y_j \xi_2, \eta_2 \rangle$ must all be 0. Again, since this holds for all $\xi_2, \eta_2 \in \mathcal{H}_2$, it follows that each $y_j = 0 \in B(\mathcal{H}_2)$, which finishes the proof. □

Example 1.14. *Let $\mathcal{H}$ be a Hilbert space and fix $n \in \mathbb{N}$. Note that since $M_n(\mathbb{C}) = B(\mathbb{C}^n)$, the previous proposition implies that $B(\mathcal{H}) \odot M_n(\mathbb{C})$ embeds into $B(\mathcal{H} \otimes \mathbb{C}^n)$, which is equal to $B(\mathcal{H}^n)$ by Exercise 6. This embedding is very natural when identify $B(\mathcal{H}) \odot M_n(\mathbb{C})$ with $M_n(B(\mathcal{H}))$ via Example 1.6. Indeed, under this identification we have*

$$\sum_{1 \le i,j \le n} x_{i,j} \odot E_{i,j} = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix},$$

and the element on the right naturally acts on vectors in $\mathcal{H}^n$ via the usual matrix action:

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n x_{1j} \xi_j \\ \vdots \\ \sum_{j=1}^n x_{nj} \xi_j \end{pmatrix} \qquad \xi_1, \ldots, \xi_n \in \mathcal{H}.$$

In particular, this defines a bounded operator with

$$\left\| \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \right\| \le \left(\sum_{1 \le i,j \le n} \|x_{i,j}\|^2 \right)^{1/2}$$

*(**exercise:** check this).*

Exercise 7. *Show that $M_n(B(\mathcal{H})) = B(\mathcal{H}^n)$. [**Hint:** how would you do this for $\mathcal{H} = \mathbb{C}$?]*

From the embedding $B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ we get tensor products of representations.

Proposition 1.15. *Given two representations $\pi_i : A_i \to B(\mathcal{H}_i)$, $i = 1, 2$, there is an induced representation*

$$\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

such that $\pi_1 \odot \pi_2(a_1 \odot a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ for all $a_i \in A_i$, $i = 1, 2$.

We have discussed extending pairs of linear maps to tensor products, but what about restricting maps on tensor products to the tensor factors? Given a $*$-homomorphism on an algebraic tensor product of C*-algebras $\phi : A \odot B \to C$, when can we define restrictions $\phi|_A : A \to C$ and $\phi|_B : B \to C$? In general this is not so easy. In the unital setting, there is a natural way to do this.

Exercise 8. *Suppose A, B, and C are C*-algebras with A and B unital and $\phi : A \odot B \to C$ a $*$-homomorphism. Then there exist $*$-homomorphisms $\phi_A : A \to C$ and $\phi_B : B \to C$ with commuting ranges such that $\phi = \phi_A \times \phi_B$.*

A little harder to prove is the following (without the assumption that A and B are unital). See [2, Theorem 3.6.2].

Theorem 1.16. *Let A and B be C*-algebras and $\pi : A \odot B \to B(\mathcal{H})$ a nondegenerate $*$-homomorphism. Then there exist nondegenerate representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ so that $\pi = \pi_A \times \pi_B$.*

Exercise 9. *How would you define the representations when A_1 and A_2 are unital? Given a representation $\pi : A_1 \odot A_2 \to B(\mathcal{H})$, show that the restrictions $\pi_i : A_i \to B(\mathcal{H})$ have commuting images.*

2. Tensor Products of C*-algebras

One of the most important constructions in C*-algebras is the tensor product. Given two C*-algebras A and B, we form a C*-tensor product $A \otimes_\alpha B$ by taking the $*$-algebraic tensor product $A \odot B$ and completing with some C*-norm. In this section, we consider the two most prominent ones. This section is taken heavily from the first half of [2, Chapter 3].

One word on notation. Because there is so much significance to the norm on a given tensor product, we will denote algebraic tensor products by $\odot$ and tensor products that are also complete with respect to a norm by $\otimes$ (possibly with decoration to denote which norm). Sometimes $\otimes$ is used in the literature to denote an algebraic tensor product, and sometimes it is used to indicate the normed tensor product space with the spatial tensor product norm Definition 2.4. Usually authors are good about warning you of this.

2.1 C*-norms on tensor products For C*-algebras A and B, $A \odot B$ is a $*$-algebra. In order to turn it into a C*-algebra, we need to be able to define a C*-norm $\|\cdot\|$ on $A \odot B$. With this, $(A \odot B, \|\cdot\|)$ will be a *pre-C*-algebra*, i.e. its completion is a C*-algebra. Much like the situation with groups, we are guaranteed the following:

- C*-norms on algebraic tensor products of C*-algebras always exist;
- there can be (very) many different C*-norms on a given algebraic tensor product of two C*-algebras;
- but we know how to describe the largest;
- and we have a nice canonical spatial norm (which unlike for groups is even the smallest!)[2]; and
- it is extremely interesting to ask when the two coincide (and this is related to the notion of amenability for groups and nuclearity for maps because math is beautiful).

Definition 2.1. *For C*-algebras A and B, a* cross norm *on a $A \odot B$ is a norm $\|\cdot\|$ such that for simple tensors we have $\|a \otimes b\| = \|a\|\|b\|$ for every $a \in A$ and $b \in B$.*

Example 2.2. *We verified in Proposition 1.12 that for $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$, the norm on $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ inherited from $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a cross norm. In fact as a consequence of Takesaki's theorem[3] (which we will discuss more later in this section) every C*-norm on $A \odot B$ is a cross norm. We will take this as a fact as we proceed.*

In Exercise 1, we saw that there is an algebraic $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \cong \mathrm{M}_n(A)$. The latter being a C*-algebra with norm induced by the norm of A in the following sense:

Recall from Exercise 7 that $\mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$ for any Hilbert space $\mathcal{H}$. Now (using the Gelfand-Naimark Theorem), we faithfully represent A on some Hilbert space $\mathcal{H}$ with an injective $*$-homomorphism $\pi : A \to B(\mathcal{H})$. This induces a $*$-homomorphism $\pi^{(n)} : \mathrm{M}_n(A) \to \mathrm{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$, which is also injective (check). Then we can define a norm on $\mathrm{M}_n(A)$ by $\|[a_{ij}]\| := \|\pi^{(n)}([a_{ij}])\|$ (injectivity implies this is a norm and not just a semi-norm), which will satisfy the C*-identity (because $(\pi^{(n)})^{-1} : \pi^{(n)}(\mathrm{M}_n(A)) \to \mathrm{M}_n(A)$ is a $*$-homomorphism).

Now pulling back the norm along this $*$-isomorphism gives a C*-norm on $\mathrm{M}_n(\mathbb{C}) \odot A$ (i.e. $\|[\lambda_{ij}] \odot a\| = \|[\lambda_{ij} a]\|$). Moreover, $\mathrm{M}_n(\mathbb{C}) \odot A$ is already complete with respect to this norm, which means it is a C*-algebra. Hence any other C*-norm we define on $\mathrm{M}_n(A)$ agrees with this norm.[4] That means we have proved the following proposition.

Proposition 2.3. *Let A be a C*-algebra and $1 \leq n < \infty$. Then there is a unique C*-norm on the algebraic tensor product $\mathrm{M}_n(\mathbb{C}) \odot A$, which comes from the $*$-isomorphism $\mathrm{M}_n(\mathbb{C}) \odot A \cong \mathrm{M}_n(A)$. Hence we write $\mathrm{M}_n(\mathbb{C}) \otimes A$.*

---

[2]This is a deep theorem due to Takesaki.

[3]Full disclosure, using this theorem is wayyyy overkill. A functional calculus argument could prove this, (see [2, Lemma 3.4.10]) but this section is already long enough.

[4]Recall that this follows from the fact that the norm on a C*-algebra is completely determined by its algebraic structure: $\|x\| = (\|x^*x\|)^{1/2} = (r(x^*x))^{1/2}$.

This identification also introduces very convenient notation, e.g. for the diagonal matrix in $\mathrm{M}_n(A)$ with $a \in A$ down the diagonal:

$$I_n \otimes a \;\leftrightarrow\; \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a \end{bmatrix}.$$

For general C*-algebras A and B, it should not be taken for granted that a C*-norm exists at all on $A \odot B$. However, it turns out the two most natural candidates both yield C*-norms.

The first is the spatial norm, i.e. the norm inherited as a subspace of bounded operators on a tensor product of Hilbert spaces. Recall that as a consequence of the GNS construction, every C*-algebra has at least one faithful representation on some Hilbert space.

Definition 2.4 (Spatial Norm). *Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. The* spatial *norm on $A_1 \odot A_2$ is*

$$\left\| \sum a_i \odot b_i \right\|_{\min} = \left\| \sum \pi_1(a_i) \otimes \pi_2(b_i) \right\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)}.$$

We will explain the $\|\cdot\|_{\min}$ notation later with Takesaki's theorem.

Exercise 10. *Check that $\|\cdot\|_{\min}$ is a semi-norm satisfying the* C*-*identity.*

Proposition 2.5. *The semi-norm $\|\cdot\|_{\min}$ is a norm, i.e. for each $x \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then $x = 0$.*

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be faithful representations. Then the algebraic tensor product map $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ is injective. By Proposition 1.13, we can view $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ as a $*$-subalgebra of $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, and consequently have $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ injective. Then for any $x = \sum_{i=1}^n a_i \odot b_i \in A_1 \odot A_2$, if $\|x\|_{\min} = 0$, then

$$0 = \|x\|_{\min} = \|\sum_{i=1}^n \pi_1(a_i) \otimes \pi_2(b_i)\| = \|(\pi_1 \odot \pi_2)(x)\|,$$

which by injectivity means $x = 0$. □

Hence $\|\cdot\|_{\min}$ is a norm, and we can define the C*-algebra

$$A \otimes B := \overline{A \odot B}^{\|\cdot\|_{\min}}.$$

It is sometimes denoted $A \otimes_{\min} B$, but we choose the undecorated notation to match the literature. In most cases this the unofficial "default" norm to take on a tensor product of C*-algebras.[5]

For a sense of perspective, dropping the representation notation, we view $A_1 \subset B(\mathcal{H}_1)$ and $A_2 \subset B(\mathcal{H}_2)$. Then there is a natural way to stick them into a common C*-algebra, i.e. $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, from whence they can inherit the C*-norm, i.e. $A_1 \otimes A_2$ is the closure of the $*$-subalgebra $A_1 \odot A_2 \subset B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

However, the norm was defined with an arbitrary choice of faithful representations. Fortunately, the value of the norm is independent of that choice.

Proposition 2.6. *Given faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$ and $\pi_i' : A_i \to B(\mathcal{H}_i')$, then the minimal tensor norms $\|\cdot\|_{\min}$ and $\|\cdot\|_{\min}'$ defined by each pair of faithful representations agree.*

The proof is nice to see because it highlights two useful techniques. The first, yet again, is approximate identities. The second is the fact that there is only one C*-norm on $\mathrm{M}_n(B)$ for any C*-algebra B.

In our proof, we limit ourselves to the countable setting to avoid the extra notation involved with nets.

Proof. By symmetry, it suffices to prove the case where $\mathcal{H}_1 = \mathcal{H}_1'$ and $\pi_1 = \pi_1'$.

We first consider the case where $A_1 = \mathrm{M}_n(\mathbb{C})$ for some n. Since both $\|\cdot\|_{\min}$ and $\|\cdot\|_{\min}'$ are C*-norms, by Proposition 2.3, for every $x = \sum_{i=1}^m T_i \odot a_i \in \mathrm{M}_n(\mathbb{C}) \odot A_2$,

$$\left\| \sum_{i=1}^n \pi_1(T_i) \otimes \pi_2(a_i) \right\| = \|x\|_{\min} = \|x\|_{\min}' = \left\| \sum_{i=1}^n \pi_1(T_i) \otimes \pi_2'(a_i) \right\|. \tag{2.1}$$

[5] For groups, it's the other way around and the maximal C*-completion of the group algebra is often the undecorated one.

Now, for the general separable case, take an increasing net of finite-rank projections $P_1 \leq P_2 \leq \dots$ in $B(\mathcal{H}_1)$ where the rank of P_n is n and such that $\|P_n\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}_1$ (i.e. P_n converge in SOT to $1_{\mathcal{H}_1}$). Then for every $T \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})$ converges in $*$-SOT[6] to T, and so we have (check)

$$\|T\| = \sup_n \|(P_n \otimes 1_{\mathcal{H}_2})T(P_n \otimes 1_{\mathcal{H}_2})\|.$$

That means for any $x = \sum_{i=1}^m a_i \odot b_i \in A_1 \odot A_2$,

$$\|x\|_{\min} = \sup_n \left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi_2(b_i) \right\|$$
$$\|x\|'_{\min} = \sup_n \left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi'_2(b_i) \right\|.$$

For $n \geq 1$, define a $*$-isomorphism $\phi : \mathrm{M}_n(\mathbb{C}) \to P_n B(\mathcal{H}) P_n$. Since ϕ is a faithful representation of $\mathrm{M}_n(\mathbb{C})$, by (2.1), we have

$$\begin{aligned} \left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi_2(b_i) \right\| &= \left\| \sum_{i=1}^m \phi(\phi^{-1}(P_n \pi(a_i) P_n)) \otimes \pi_2(b_i) \right\| \\ &= \left\| \sum_{i=1}^m \phi(\phi^{-1}(P_n \pi(a_i) P_n)) \otimes \pi'_2(b_i) \right\| \\ &= \left\| \sum_{i=1}^m P_n \pi(a_i) P_n \otimes \pi'_2(b_i) \right\|. \end{aligned}$$

It follows that $\|x\|_{\min} = \|x\|'_{\min}$. □

So, given C*-algebras A_1 and A_2 and faithful nondegenerate representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we complete $\pi_1 \odot \pi_2$ to a faithful representation

$$\pi_1 \otimes \pi_2 : A_1 \otimes A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

There is another often useful description of the minimal tensor norm.

Proposition 2.7. *For* C**-algebras* A_1 *and* A_2*, and* $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$,

$$\|x\|_{\min} = \sup\{\| \sum_{j=1}^n \pi_1(a_j) \otimes \pi_2(b_j)\| \ : \ \pi_i : A_i \to B(\mathcal{H}_i) \ \textit{(nondegenerate) representations}\}.$$

Proof. Let $\pi_i : A_i \to B(\mathcal{H}_i)$ be representations and $\sigma_i : A_i \to B(\mathcal{H}'_i)$ be faithful representations. Then $\pi_i \oplus \sigma_i : A_i \to B(\mathcal{H}_i \oplus \mathcal{H}'_i)$ is a faithful representation. Let $P_i \in B(\mathcal{H}_i \oplus \mathcal{H}'_i)$ be the compression to $\mathcal{H}_i$ for each $i = 1, 2...$ □

Exercise 11. *Finish the proof of Proposition 2.7. This is an example of a technique where one can* dilate *a map to one with a desired property (e.g. faithfulness) and then* cut down *to the original map to draw the desired conclusion.*

Just as with groups, there is another natural norm which comes from taking all possible representations.

Definition 2.8 (Maximal Norm)**.** *Let* A *and* B *be* C**-algebras. We define the maximal* C**-tensor norm on* $A \odot B$ *by*

$$\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}) \ \textit{a (nondegenerate) rep}\}$$

for all $x \in A \odot B$.

[6] $S_n \to S$ in $*$-SOT if $S_n \to S$ in SOT and $S_n^* \to S^*$ in SOT.

The first question is if this is even finite; it is by Theorem 1.16. Indeed, given $\pi : A \odot B \to B(\mathcal{H})$, with restrictions $\pi|_A$ and $\pi|_B$, then we have for all simple tensors $a \odot b \in A \odot B$,

$$\|\pi(a \odot b)\| = \|\pi|_A(a)\pi|_B(b)\| \leq \|\pi|_A(a)\|\|\pi|_B(b)\| \leq \|a\|\|b\|. < \infty.$$

Just as one argues for universal/ full group C*-algebras, this with the triangle inequality guarantees that $\|x\|_{\max} < \infty$ for all $x \in A \odot B$.

Exercise 12. *Check that $\|\cdot\|_{\max}$ is a semi-norm satisfying the* C*-*identity.*

For any pair of faithful representations $\pi_i : A_i \to B(\mathcal{H}_i)$, we get a representation $\pi = \pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It follows that for any $x \in A_1 \odot A_2$,

$$\|x\|_{\min} = \|\pi(x)\| \leq \|x\|_{\max}.$$

So, for any $x \in A_1 \odot A_2$,

$$\|x\|_{\max} = 0 \Rightarrow \|x\|_{\min} = 0 \Rightarrow x = 0,$$

which means $\|\cdot\|_{\max}$ is a norm. Hence we define the C*-algebra

$$A_1 \otimes_{\max} A_2 := \overline{A_1 \odot A_2}^{\|\cdot\|_{\max}}.$$

Remark 2.9. *Note that by definition, the $*$-algebra $A_1 \odot A_2$ is a dense subalgebra in $A_1 \otimes_{\max} A_2$ and $A_1 \otimes A_2$.*

Just as with groups, the maximal tensor product enjoys a universal property.

Proposition 2.10. *If $\phi : A_1 \odot A_2 \to C$ is a $*$-homomorphism, then there exists a unique $*$-homomorphism $A_1 \otimes_{\max} A_2 \to C$, which extends ϕ. In particular, any pair of $*$-homomorphisms $\phi_i : A_i \to C$ with commuting ranges induces a unique $*$-homomorphism*

$$\phi_1 \times \phi_2 : A \otimes_{\max} B \to C.$$

Note that this is really just a statement about norms, and it is a theme we've seen before (e.g. universal/ full group C*-algebras). Let's flesh out a more general idea that underlies both.

Suppose B and C are C*-algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. (Notice that, unless $B_0 = B$, this means B_0 is *not* a C*-algebra.) The only obstruction to extending π to a $*$-homorphism on B is if π is not contractive on B_0, i.e. $\|\pi(b)\| > \|b\|$ for some $b \in B_0$. In other words, π extends to B iff π is contractive on B_0. The necessity is easy to see. Indeed, if π does extend to B, then the C*-norm on B forces π to be contractive on all of B, including B_0. On the other hand, if $\pi : B_0 \to C$ is a contractive $*$-homomorphism, then it is in particular bounded, which means it extends to a contractive homomorphism $\pi : B \to C$. Moreover, for any $b \in B$ with $b_n \in B_0$ converging to b, we have $\pi(b_n) \to \pi(b)$ and hence $\pi(b_n)^* \to \pi(b)^*$. Then by uniqueness of limits, $\pi(b^*) = \pi(b)^*$ since

$$\|\pi(b_n)^* - \pi(b^*)\| = \|\pi(b_n^*) - \pi(b^*)\| \to 0.$$

For the sake of reference, we record this in a proposition:

Proposition 2.11. *Suppose B and C are* C*-*algebras, $B_0 \subset B$ is a dense $*$-subalgebra, and $\pi : B_0 \to C$ is a $*$-homomorphism. Then π extends to B iff π is contractive on B_0.*

With that digression, the proof of proposition 2.10 is quite immediate.

Proof of Proposition 2.10. Take a faithful nondegenerate representation $\pi : C \to B(\mathcal{H})$. Then $\pi \circ \phi : A_1 \odot A_2 \to B(\mathcal{H})$ is a contractive $*$-homomorphism (with respect to the $\|\cdot\|_{\max}$ norm) and hence extends to $A \otimes_{\max} A_2$. □

It follows from this that $\|\cdot\|_{\max}$ is the largest possible C*-norm on $A_1 \odot A_2$.

Corollary 2.12. *Given any* C*-*norm $\|\cdot\|$ on $A_1 \odot A_2$, there is a surjective $*$-homomorphism $A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ extending the identity map on $A_1 \odot A_2$.*

Proof. Suppose $\|\cdot\|$ is another C*-norm on $A_1 \odot A_2$. Then the identity map $A_1 \odot A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}$ is a $*$-homomorphism, which then extends to a $*$-homorphism

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|}.$$

Since it is a $*$-homomorphism, its image is closed and contains the dense subset $A_1 \odot A_2$, and so it is a surjection. As a surjective $*$-homomorphism, it is contractive, and so $\|x\|_{\max} \geq \|x\|$ for all $x \in A_1 \odot A_2$. □

Remark 2.13. *Very often in the literature, the closure of $A \odot B$ with respect to an arbitrary tensor norm is denoted by $A \otimes_\alpha B$ where the norm is denoted by $\|\cdot\|_\alpha$.*

It turns out that the spatial norm $\|\cdot\|_{\min}$ *is* the minimal C*-norm on $A_1 \odot A_2$. This is an important theorem due to Takesaki whose proof involves some heavy work in extending states to tensor products. For the sake of time, we will have to take this for granted. The proof is worked out in [2, Section 3].

Theorem 2.14 (Takesaki)**.** *The spatial norm $\|\cdot\|_{\min}$ is the minimal* C*-*norm on $A_1 \odot A_2$. In other words, given any* C*-*norm $\|\cdot\|$ on $A_1 \odot A_2$, there are surjective ∗-homomorphisms*

$$A_1 \otimes_{\max} A_2 \to \overline{A_1 \odot A_2}^{\|\cdot\|} \to A_1 \otimes A_2$$

extending the identity map

$$A_1 \odot A_2 \to A_1 \odot A_2 \to A_1 \odot A_2.$$

It follows that if the *natural* surjection $A_1 \otimes_{\max} A_2 \to A_1 \otimes A_2$ is injective, then $A_1 \odot A_2$ has a unique tensor norm. This fact is often indicated by writing

$$A_1 \otimes_{\max} A_2 = A_1 \otimes A_2.$$

Remark 2.15. *It is important here that it is this natural surjection that is also injective, i.e. the one that extends the identity map $A_1 \odot A_2$.*

We have been avoiding the non-unital elephant in the room. We relegate the proof to [2, Corollary 3.3.12].

Proposition 2.16. *If A and B are* C*-*algebras with A non-unital, then any* C*-*norm on $A \odot B$ can be extended to a* C*-*norm on $\tilde{A} \odot B$ (meaning the norms agree on $A \odot B \subset \tilde{A} \odot B$). Similarly, when both A and B are non-unital, any* C*-*norm can be extended to $\tilde{A} \odot \tilde{B}$.*[7]

Exercise 13. *For* C*-*algebras A and B, we have canonical*[8] *isomorphisms $A \otimes B \cong B \otimes A$ and $A \otimes_{\max} B \cong B \otimes_{\max} A$.*

Remark 2.17 (Remark on tensors and commutivity)**.** *Given* C*-*algebras A_1 and A_1, an example of a representation of $A_1 \odot A_2 \to B(\mathcal{H})$ is the tensor product of two representations,*

$$\sigma_1 \odot \sigma_2 : A_1 \odot A_2 \to B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \subset B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

But in general, there can be many representations that are not of this form, i.e. for some $x \in A_1 \odot A_2$, we could have

$$\begin{aligned} \|x\|_{\max} &= \sup\{\|\pi(x)\| \ : \pi : A_1 \odot A_2 \to B(\mathcal{H})\} \\ &> \sup\{\|\pi_1 \odot \pi_2(x)\| \ : \pi_i : A_i \to B(\mathcal{H}_i)\}. \end{aligned}$$

On an philosophical level, this is a question about commutivity. Given C*-*algebras A_1 and A_2, is there any context (=* C*-*algebra they can be simultaneously embedded into) where A_1 and A_2 commute but* not *as tensors. Let's try to flesh this out a little.*

Given a representation $\pi : A_1 \odot A_2 \to B(\mathcal{H})$, the restrictions $\pi_i : A_i \to B(\mathcal{H})$ have commuting images (Exercise 9). When $\pi = \sigma_1 \odot \sigma_2 : A_1 \odot A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have a much better idea of what the images are and why they commute. In this case the restrictions are given for $a_i \in A_i$ by

$$\pi_1(a_1) = \sigma_1(a_1) \otimes 1_{\mathcal{H}_1} \quad \text{and} \quad \pi_2(a_2) = 1_{\mathcal{H}_2} \otimes \sigma_2(a_2).$$

Then we have

$$\pi_1(a_1)\pi_2(a_2) = \big(\sigma_1(a_1) \otimes 1_{\mathcal{H}_1}\big)\big(1_{\mathcal{H}_2} \otimes \sigma_2(a_2)\big) = \sigma_1(a_1) \otimes \sigma_2(a_2) = \big(1_{\mathcal{H}_2} \otimes \sigma_2(a_2)\big)\big(\sigma_1(a_1) \otimes 1_{\mathcal{H}_1}\big) = \pi_2(a_2)\pi_1(a_1).$$

Sometimes the maximum and minimum norms on a tensor product $A \odot B$ do coincide, e.g., if $A = \mathrm{M}_n(\mathbb{C})$.

[7] In general (i.e. when we don't have $A = \tilde{A}$ or $B = \tilde{B}$, this is a larger algebra than $\widetilde{A \odot B}$.

[8] i.e. This is another way of saying "natural". In this setting, this means the maps extend the usual algebraic maps.

Example 2.18. *Let $\mathcal{K}$ denote the compact operators on some Hilbert space $\mathcal{H}$ and A any* C*-*algebra.*

First we claim that $FR(\mathcal{H}) \odot A$ is a dense $*$-subalgebra of $\mathcal{K} \odot A$ with respect to any C*-*norm on $\mathcal{K} \odot A$. Recall that that $FR(\mathcal{H})$ is dense in $\mathcal{K}$. Now, suppose $S \odot a \in \mathcal{K} \odot A$ and $S_j \in FR(\mathcal{H})$ a sequence with $S_j \to S$. Recall that any* C*-*norm $\|\cdot\|$ on $\mathcal{K} \odot A$ is a cross norm, and so for any* C*-*norm $\|\cdot\|$ on $\mathcal{K} \odot A$, we have*

$$\|(S \odot a) - (S_j \odot a)\| = \|(S - S_j) \odot a\| = \|S - S_j\|\|a\| \to 0.$$

Using the triangle inequality, we can extend this to show that any $x = \sum_{j=1}^m T_j \odot a_j \in \mathcal{K} \odot A$ can be approximated in any C*-*norm by sums of simple tensors of finite rank operators.*

So if we know $\|x\|_{\max} = \|x\|_{\min}$ for any $x \in FR(\mathcal{H}) \odot A$, then it follows that the natural surjection $\mathcal{K} \otimes_{\max} A \to \mathcal{K} \otimes A$ is isometric and $\mathcal{K}$ is nuclear. Fix an arbitrary $x = \sum_{j=1}^m T_j \odot a_j \in FR(\mathcal{H}) \odot A$, and let $\pi : \mathcal{K} \odot A \to B(\mathcal{H})$ be a representation. Then there exists a projection $P \in B(\mathcal{H})$ of rank $n < \infty$ such that $T_j = PT_jP$ for all j, and $x = \sum_{j=1}^m PT_jP \odot a_j$. Hence $x \in PB(\mathcal{H})P \odot A$. From Exercise 7.41 from Day 1 Lectures, we have a $*$-isomorphism $\phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P$, and hence a representation $\pi' := \pi \circ (\phi \odot id_A) : \mathrm{M}_n(\mathbb{C}) \odot A \to B(\mathcal{H})$.

Since we know $\mathrm{M}_n(\mathbb{C}) \otimes_{\max} A = \mathrm{M}_n(\mathbb{C}) \otimes_{\min} A$, we know that for any faithful representations $\sigma_1 : \mathrm{M}_n(\mathbb{C}) \to B(\mathcal{H}_1)$ and $\sigma_2 : A \to B(\mathcal{H}_2)$,

$$\begin{aligned}\|\sum_{j=1}^m \sigma_1(\phi^{-1}(PT_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H}_1 \otimes \mathcal{H}_2)} &= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\min} \\ &= \|\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j\|_{\max} \geq \|\pi'(\sum_{j=1}^m \phi^{-1}(PT_jP) \odot a_j)\| \\ &= \|\pi(\sum_{j=1}^m PT_jP \odot a_j)\| = \|\pi(x)\|.\end{aligned}$$

In particular, this holds for the faithful representations $\sigma_1 = \mathrm{id}_{\mathcal{K}} \circ \phi : \mathrm{M}_n(\mathbb{C}) \to PB(\mathcal{H})P \subset \mathcal{K} \hookrightarrow B(\mathcal{H})$ and any faithful representation σ_2 of A. But then we have

$$\begin{aligned}\|x\|_{\min} &= \|\sum_{j=1}^m \mathrm{id}_{\mathcal{K}}(T_j) \odot \sigma_2(a_j)\|_{B(\mathcal{H} \otimes \mathcal{H}_2)} \\ &= \|\sum_{j=1}^m \sigma_1(\phi^{-1}(PT_jP)) \odot \sigma_2(a_j)\|_{B(\mathcal{H} \otimes \mathcal{H}_2)} \\ &\geq \|\pi(x)\|.\end{aligned}$$

Since $\pi : \mathcal{K} \odot A \to B(\mathcal{H})$ was arbitrary, it follows that

$$\|x\|_{\min} \geq \|x\|_{\max},$$

which finishes the proof.

2.2 Continuous linear maps on tensor products In Takesaki's proof that $\|\cdot\|_{\min}$ is the smallest C*-norm, a delicate and crucial part of the argument is showing that states extend to tensor products, i.e. for $\phi_i \in S(A_i)$, $\phi_1 \odot \phi_2$ extends to a state on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ for any C*-norm $\|\cdot\|$ (mapping into $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$).

Given a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, we have a $*$-homomorphism

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

defined on the dense $*$-subalgebra $A_1 \odot A_2$ of $\overline{A_1 \odot A_2}^{\|\cdot\|}$ where $\|\cdot\|$ is any C*-norm. By Proposition 2.11, this extends to a $*$-homomorphism on $\overline{A_1 \odot A_2}^{\|\cdot\|}$ iff $\phi_1 \odot \phi_2$ is contractive on sums of simple tensors. Naturally, this depends on the norm we put on $B_1 \odot B_2$ (e.g. if $A_i = B_i$ and we give $A_1 \odot A_2$ the maximal norm and $B_1 \odot B_2$ the minimal norm).

Let us see how this works with respect to the minimal tensor product norms.

Corollary 2.19. *For a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism*

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2.$$

Proof. We are charged with showing that $\phi_1 \odot \phi_2$ is continuous with respect to the topologies on $A_1 \odot A_2$ and $B_1 \odot B_2$ induced by their respective $\|\cdot\|_{\min}$ norms. We know that there exist faithful representations $\pi_i^A : A_i \to B(\mathcal{H}_i^A)$ and faithful representations $\pi_i^B : B_i \to B(\mathcal{H}_i^B)$. So if $x = \sum_{j=1}^n a_j \odot b_j \in A_1 \odot A_2$, the fact that $*$-homomorphisms are norm-decreasing means that

$$\|x\|_{A_1 \otimes_{\min} A_2} = \| \sum_{j=1}^n \pi_1^A(a_j) \otimes \pi_2^A(b_j)\| \geq \| \sum_{j=1}^n \pi_1^B(\phi_1(a_j)) \otimes \pi_2^B(\phi_2(b_j))\| = \|\phi_1 \odot \phi_2(x)\|_{B_1 \otimes_{\min} B_2}.$$

Then we are done by Proposition 2.11. (Or alternatively using the fact that each $\pi_i^B \phi_i : A_i \to B(\mathcal{H}_i^B)$ is a representation of A_i and appealing to Proposition 2.7.) □

Exercise 14. *Show that for a pair of $*$-homomorphisms $\phi_i : A_i \to B_i$, the algebraic tensor product $\phi_1 \odot \phi_2$ extends to a $*$-homomorphism on*

$$\phi_1 \otimes_{\max,\beta} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_\beta B_2$$

for any C*-*tensor product $B_1 \otimes_\beta B_2$. Hint: faithfully represent $B_1 \otimes_\beta B_2 \subset B(\mathcal{H})$. Then $\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B(\mathcal{H})$ is a $*$-homomorphism, which extends to $A_1 \otimes_{\max} A_2$.*

However, many maps that we want to work with (e.g. states) are not necessarily $*$-homomorphisms. Hence it is important to understand which class of bounded linear maps extend to tensor products, in particular, for which bounded linear maps $\phi_i : A_i \to B_i$ does $\phi_1 \odot \phi_2$ extend to continuous linear maps

$$\phi_1 \otimes_{\max} \phi_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2$$

and

$$\phi_1 \otimes_{\min} \phi_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2?$$

Let us consider an example where this fails.

Example 2.20. *Consider $\mathcal{K} = \mathcal{K}(\ell^2)$. It follows from Example 2.18 that the completion of $\mathcal{K} \odot \mathcal{K}$ under any tensor norm can be identified with the completion of $\mathcal{K} \odot \mathcal{K}$ with respect to the norm on $B(\ell^2 \otimes \ell^2)$ (via the tensor product of faithful representations $id_{\mathcal{K}} \odot id_{\mathcal{K}}$). For each i, j, we define the rank one operator $P_{i,j} = \langle \cdot, e_j\rangle e_i$. (Think of these as an infinite-dimensional version of the matrix units for $M_n(\mathbb{C})$.) For each $n \geq 1$, define $V_n \in \mathcal{K} \otimes \mathcal{K}$ by*

$$V_n := \sum_{i,j=1}^n P_{i,j} \otimes P_{j,i}.$$

Then V_n is a partial isometry. (Indeed, since $P_{i,j}P_{l,k} = \delta_{j,l}P_{i,k}$, we can compute that $V_n^* V_n = P_n \odot P_n$ where P_n is the rank n projection sending $e_j \mapsto e_j$ for $1 \leq j \leq n$ and $e_j \mapsto 0$ for $j > n$.) So $\|V_n\| = 1$ for all n.

Now considering each $T = [t_{ij}] \in \mathcal{K}$ as an array, we let $Tr : \mathcal{K} \to \mathcal{K}$ denote the transpose map, which is given by $Tr([t_{ij}]) = [t_{ji}]$. This is a linear $*$-preserving isometric map (since $T^* = [\bar{t}_{ji}]$), and

$$Tr \odot 1_{\mathcal{K}}(V_n) = \sum_{i,j=1}^n e_{ji} \otimes e_{ji}.$$

Now, consider the vector $\xi = \sum_{k=1}^n e_k \otimes e_k$. One computes

$$\begin{aligned}\|Tr \odot 1_{\mathcal{K}}(V_n)\xi\| &= \| \sum_{i,j=1}^n \sum_{k=1}^n \langle e_k, e_j\rangle e_i \otimes \langle e_k, e_j\rangle e_i\| \\ &= \| \sum_{i=1}^n \sum_{k=1}^n \langle e_k, e_k\rangle e_i \otimes \langle e_k, e_k\rangle e_i\| \\ &= \| \sum_{i=1}^n n(e_i \otimes e_i)\| = \|n\xi\| = n\|\xi\|.\end{aligned}$$

In particular, this means that $\|Tr \odot 1_{\mathcal{K}}(V_n)\| \geq n$ and hence $\|Tr \odot 1_{\mathcal{K}}\| \geq n$ for all $n \in \mathbb{N}$. This is an unbounded operator and hence not continuous.

So what kinds of bounded linear maps on C*-algebras yield continuous tensor product maps? Notice that the above example is $*$-preserving, so that's not enough. We have remarked several times that much of the structure of the C*-algebra is preserved by positive elements. Perhaps we need to consider linear maps $\phi : A \to B$ that send positive elements in A to positive elements in B? But even that isn't enough. It turns out that the transpose map above does send positive elements to positive elements. So, what gives? This is where we finally motivate the idea of *completely* positive maps.

Definition 2.21. *A linear map $\phi : A \to B$ between* C*-*algebras is* positive *if $\phi(a) \geq 0$ for all $a \in A_+$.* completely positive *if its matrix amplification*

$$\phi^{(n)} : \mathrm{M}_n(\mathbb{C}) \otimes A \to \mathrm{M}_n(\mathbb{C}) \otimes B$$

is positive for all $n \geq 1$.

We can also define (completely) positive maps on operator subsystems *of unital* C*-*algebras. Given a unital* C*-*algebra A, an* operator system *(also known as operator subsystem) is a unital (closed) self-adjoint subspace $1_A \in X \subset A$. A linear map $\phi : X \to B$ is (completely) positive if it satisfies the above defnitions on elements in X.*

We abbreviate completely positive as "c.p.".

Remark 2.22. *One can show that a c.p. map $\varphi : X \to B$ is* completely bounded *meaning* $\sup_n \|\varphi^{(n)}\| < \infty$. *In fact, it turns out* $\sup_n \|\varphi^{(n)}\| = \|\varphi\| = \|\varphi(1_A)\|$. *When it is contractive, we abbreviate it as "c.p.c." (or sometimes "c.c.p."), and when it is unital (and hence contractive by the above), we write "u.c.p.".*

Here's an important class of examples.

Example 2.23. *Let $\psi : A \to B$ be a cp map between* C*-*algebras and $b \in B$. Then the map $\phi := b^*\psi(\cdot)b : A \to B$ is linear and positive (exercise). It is moreover completely positive. Indeed, for each $n \geq 1$ and positive element $[a_{ij}] \in \mathrm{M}_n(A)$,*

$$\phi^{(n)}([a_{ij}]) = \begin{bmatrix} b^*\phi(a_{11})b & \dots & b^*\phi(a_{1n})b \\ \vdots & \ddots & \vdots \\ b^*\phi(a_{n1})b & \dots & b^*\phi(a_{nn})b \end{bmatrix} = \begin{bmatrix} b^* & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b^* \end{bmatrix} \begin{bmatrix} \phi(a_{11}) & \dots & \phi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(a_{n1}) & \dots & \phi(a_{nn}) \end{bmatrix} \begin{bmatrix} b & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b \end{bmatrix}.$$

*Observe (**exercise**) that when $\|b\| \leq 1$, ϕ is completely positive and contractive.*

Example 2.24 (Tomiyama)**.** *Another example of a c.p.c. map is a conditional expectation.*

The two biggest theorems for completely positive maps are Stinespring's Dilation Theorem and Arveson's Extension Theorem. For now, we just state the later.

Theorem 2.25 (Arveson's Extension Theorem)**.** *Let A be a unital* C*-*algebra with operator subsystem and $X \subset$ an operator subsystem. Then any c.p.c. map $\varphi : X \to B(\mathcal{H})$ extends to a c.p.c. map $\tilde{\varphi} : A \to B(\mathcal{H})$ with $\tilde{\varphi}|_X = \varphi$.*

Theorem 2.26. *Let $\phi_i : A_i \to B_i$ be linear cp maps. Then the algebraic tensor product map*

$$\phi_1 \odot \phi_2 : A_1 \odot A_2 \to B_1 \odot B_2$$

extends to a linear cp map (which is then also bounded and hence continuous) map on both the maximal and minimal tensor products:

$$\begin{aligned} \phi_1 \otimes \phi_2 :& A_1 \otimes A_2 \to B_1 \otimes B_2 \\ \phi_1 \otimes_{\max} \phi_2 :& A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2. \end{aligned}$$

Moreover, we have $\|\phi_1 \otimes_{\max} \phi_2\| = \|\phi_1 \otimes \phi_2\| = \|\phi_1\|\|\phi_2\|$.

Remember that we have already proved this for $*$-homomorphisms. Stinespring's Dilation theorem will allow us to transfer this fact to cpc maps.

In full disclosure, we need a stronger version of this to prove the $\otimes_{\max}$ part of Theorem 2.26, so we direct you to [2, Proposition 1.5.6] and its use in the proof of [2, Theorem 3.5.3]. But for the sake of seeing Stinespring's Theorem in action, let's prove that the algebraic tensor product of cp maps extends to a cp map between spatial tensor products.

Proof of Theorem 2.26 (for spatial tensor). Let A_1, A_2, B_1, B_2 be C*-algebras and $\phi_i : A_i \to B_i$ cp maps. First, by taking faithful representations, it suffices to assume that $B_i \subset B(\mathcal{H}_i)$ for $i = 1, 2$ (why?). Then $\phi_i : A_i \to B(\mathcal{H}_i)$ are cp maps, which have Stinespring dilations $(\pi_i, \mathcal{H}_i', V_i)$ for $i = 1, 2$. Since these are $*$-homomorphisms, $\pi_1 \odot \pi_2 : A_1 \odot A_2 \to B(\mathcal{H}_1') \odot B(\mathcal{H}_2') \subset B(\mathcal{H}_1' \otimes \mathcal{H}_2')$ extends to $A_1 \otimes A_2$. Define the map $\phi_1 \otimes \phi_2 : A_1 \otimes A_2 \to B_1 \otimes B_2 \subset B(\mathcal{H}_1' \otimes \mathcal{H}_2')$ by

$$\phi_1 \otimes \phi_2(x) = (V_1 \otimes V_2)^*(\pi_1 \otimes \pi_2)(x)(V_1 \otimes V_2).$$

By Example 2.23, this is a cp map. Moreover, for elementary tensors $a_1 \odot a_2 \in A_2 \odot A_2$, we have

$$\phi_1 \otimes \phi_2(a_1 \odot a_2) = (V_1^* \pi_1(a_1)V_1) \otimes (V_2^* \pi_2(a_2)V_2) = \phi_1(a_1) \odot \phi_2(a_2),$$

which means (by linearity) that $\phi_1 \otimes \phi_2|_{A_1 \odot A_2} = \phi_1 \odot \phi_2$. □

References

[1] W. Arveson, *A*n Invitation to C*-algebras. Graduate Texts in Mathematics, Springer-Verlag New York Inc., 1976.

[2] N. Brown and N. Ozawa, *C*-Algebras and Finite-Dimensional Approximations*. Graduate Studies in Mathematics. Volume 88 AMS. (2008).

[3] G. J. Murphy, C*-*algebras and Operator Theory*. Academic Press, 1990.

[4] M. Rørdam, F. Larsen, N.J. Laustsen. *A*n Introduction to *K*-Theory for C*-Algebras. London Mathematical Society Student Texts, **49**. Cambridge University Press, Cambridge, 2000.