

Question (Connes 76). (Popa)

Does every separably-acting II_1 -factor embed into some ultrapower R^ω of the hyperfinite II_1 -factor R ? facial alg YN (7)

Theorem (Kirchberg) The following conjectures are equivalent

1) CEP

2) Every C^* -algebra is QWEP (Kirchberg's QWEP conj)

3) $C^*(F)$ has WEP for any (every) non-abelian free gp

4) $C^*(F) \otimes_{\max} C^*(F) = C^*(F) \otimes_{\min} C^*(F)$ " " " " " "

6) $C^*(\mathbb{Z}_m^{*k}) \otimes_{\max} C^*(\mathbb{Z}_m^{*k}) = C^*(\mathbb{Z}_m^{*k}) \otimes_{\min} C^*(\mathbb{Z}_m^{*k})$

for each $k, m \geq 2$ with either $k > 2$ or $m > 2$

($\mathbb{Z}_m^{*k} = \langle \mathbb{Z}/m\mathbb{Z} \rangle^k$)

5) $C^*(F \times F)$ is RFD " " " " " "

7) LLP \Rightarrow WEP

Let's recap what we know:

A C^* -alg A has WEP iff $A \otimes_{\max} C^*(F) = A \otimes_{\min} C^*(F)$

for any (all) non-abelian free gp(s) F .

Hence (3) \Leftrightarrow (4)

By comparing universal properties, one can

see that $C^*(F) \otimes_{\max} C^*(F) = C^*(F \times F)$.

Moreover, the minimum tensor product of any two RFD C^* -algebras is RFD

(Indeed, take the tensor products of respective separating families of 1-dim'l reps.)

Since $C^*(F)$ is RFD, (4) \Rightarrow (5) follows.

In fact, given two RFD C^* -algebras A, B ,

$A \otimes_{\max} B$ is RFD iff $A \otimes_{\min} B = A \otimes_{\max} B$.

(Essentially b/c any rep $\pi: A \otimes B \rightarrow M_n$ extends to $A \otimes_{\min} B$)

So, we know (3) \Leftrightarrow (4) \Leftrightarrow (5). 100

Since $A \otimes_{\max} B$ and $A \otimes_{\min} B$ have sep families of f.d. reps,

want to show that for any f.d. rep $\pi: A \otimes_{\max} B \rightarrow M_n$,
 \exists rep $\pi': A \otimes_{\min} B \rightarrow M_n$ s.t. $\|\pi'(x)\| \geq \|\pi(x)\| \forall x \in A \otimes B$.

Claim Any f.d. rep $\pi: A \otimes B \rightarrow M_n$ extends to a rep of $A \otimes_{\min} B$.

Let π_A and π_B be the restrictions. $\pi = \pi_A \times \pi_B$
 $(\pi(a \otimes b) = \pi_A(a) \pi_B(b) \forall a \in A, b \in B)$

Then the natural inclusions $L_A: \pi_A(A) \hookrightarrow M_n$, $L_B: \pi_B(B) \hookrightarrow M_n$
induce $L_A \times L_B: \pi_A(A) \otimes_{\max} \pi_B(B) \rightarrow M_n$

$$\text{w/ } L_A \times L_B(\pi_A(a) \otimes \pi_B(b)) = \pi_A(a) \pi_B(b) \forall a \in A, b \in B$$

OTOTI, we have a map $\pi_A \otimes \pi_B: A \otimes_{\min} B \rightarrow \pi_A(A) \otimes_{\min} \pi_B(B)$

Since $\pi_A(A) \otimes_{\max} \pi_B(B) = \pi_A(A) \otimes_{\min} \pi_B(B)$
we can define $\tilde{\pi}: A \otimes_{\min} B \rightarrow M_n$ by $\tilde{\pi} := (L_A \times L_B) \circ (\pi_A \otimes \pi_B)$

$$\Rightarrow \tilde{\pi}(a \otimes b) = (L_A \times L_B)(\pi_A(a) \otimes \pi_B(b)) = \pi_A(a) \pi_B(b) = \pi(a \otimes b)$$

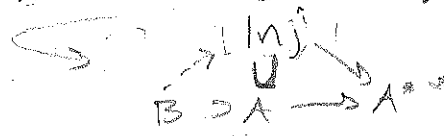
Let's talk about $(1) \Rightarrow (2) \Rightarrow (7) \Rightarrow (3) \Rightarrow (1)$
 OK. $(7) \Rightarrow (3)$ we know

(2)

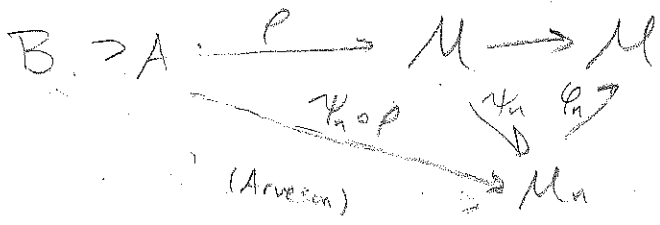
Recall For C^* -algebras $A \subset B$, a map $\psi: B \rightarrow A^{**}$ extending the canonical embedding $1_A: A \rightarrow A^{**}$ is a weak conditional expectation.

When such a ψ exists, we say $A \hookrightarrow B$ rwi.

When every embedding of A (or equiv any $A \hookrightarrow B$) is rwi, then A has WEP



Examples All injective C^* -algebras have WEP including all semidiscrete (hence also all) VN algs



$$\phi_n \circ \psi_n \rightarrow \text{id}_M \text{ pt-ultrawkly}$$

$$(\psi(\phi_n \circ \psi_n(a)) \rightarrow \psi(a) \forall a \in M_n)$$

So $\phi_n \circ \psi_n \circ \rho: B \rightarrow M$ wcp, $\phi_n \circ \psi_n \circ \rho|_A = \phi_n \circ \psi_n \circ \rho$
 and $\phi_n \circ \psi_n \circ \rho \rightarrow \rho$ pt ultrawkly

Then $\phi_n \circ \psi_n \circ \rho$ has a clusterpt in the ultrawk top (by Banach-Alaoglu)

In particular, \mathbb{K} has WEP.

Prop (Kir) WEP is closed under taking direct prod.

Proof Let $\{A_i\}_i$ be a family of WEP C^* algs w/ $A_i \subset B(\mathcal{H}_i)$. Then $\prod_i B(\mathcal{H}_i)$ is injective and so, it suffices to find $\psi: \prod_i B(\mathcal{H}_i) \rightarrow (\prod_i A_i)^{**}$ extending $\prod_i A_i \hookrightarrow (\prod_i A_i)^{**}$.

Claim For any f.d. subsp $E \subset \prod_i B(\mathcal{H}_i)$ & $\varepsilon > 0$, \exists a contractive linear map $\psi: E \rightarrow \prod_i A_i =: A$ s.t. $\|\psi \circ \text{id}_E - \text{id}_E\| < \varepsilon$.

3) Principle of Local Reflexivity

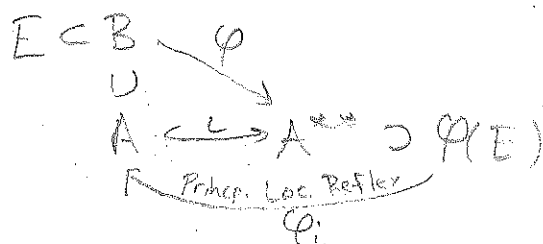
Let X be a Banach sp & $E \subset X^{**}$ f.d.m' subspace
 Then \exists a net $\{\varphi_i: E \rightarrow X\}$ of contractions converging
 to id $_E$ ptwk*, s.t. $\{\varphi_i|_{E \cap A}\}$ converges to id $_E$ pt-norm

Lemma Let $A \subset B$ be C^* -alg. TFAE

- (1) $A \hookrightarrow B$ w.i. (1') \exists Normal cond exp $B^{**} \rightarrow A^{**}$
 (2) For any f.d. $E \subset B$ & $\varepsilon > 0$, \exists contractive linear
 map $\psi: E \rightarrow A$ s.t. $\|\psi|_{E \cap A} - id_{E \cap A}\| \leq \varepsilon$.

Pf

(1) \Rightarrow (2)



set $\psi_i = \varphi_i \circ \varphi$
 Since $\varphi|_A = id$,

$$\psi_i|_{E \cap A} = \varphi_i|_{E \cap A} = \varphi_i|_{\varphi(E) \cap A} \xrightarrow{\text{ptwk}} id_{\varphi(E) \cap A} = id_{E \cap A}$$

a compactness argument \uparrow gives the uniformity

(2) \Rightarrow (1) For each $E \subset B$ f.d.m & $\varepsilon > 0$, let
 $\psi_{E, \varepsilon}: E \rightarrow A$ contraction w/ $\|\psi|_{E \cap A} - id_{E \cap A}\| \leq \varepsilon$
 Let $\varphi: B \rightarrow A^{**}$ be a wk*-cluster pt.

$\Rightarrow \varphi$ is a contractive linear map w/ $\varphi|_A = id_A$
 Then φ extends to a wk*-cts contractive
 linear map $\varphi^{**}: B^{**} \rightarrow A^{**}$ (on the level
 of Banach spaces)

Then φ^{**} is a contraction s.t. $\varphi|_{A^{**}} = id_{A^{**}}$

By Tomiyama's Theorem, φ^{**} is a
 conditional expectation & hence cpc.

By the Lemma, we know that for each $E_i \subset B(\mathcal{H}_i) \cup A_i$
 \exists a contractive linear map $\psi_i: E_i \rightarrow A_i$ s.t.

$$\|\psi_i|_{E_i \cap A_i} - \text{id}|_{E_i \cap A_i}\| < \epsilon.$$

(4)

Then $\psi := \bigoplus \psi_i: E \rightarrow \prod A_i$ is contractive w/
 $\|\psi|_{E \cap \prod A_i} - \text{id}|_{E \cap \prod A_i}\| < \epsilon$, and we are done
 by the lemma. \square

Cor $l^\infty(\mathbb{R}) = \prod_{\mathbb{N}} \mathbb{R}$ has WEP

Then for any (free) ultrafilter $\omega \in \beta \mathbb{N} \setminus \mathbb{N}$,

$$R^\omega = l^\infty(\mathbb{R}) / \{ (x_n) : \lim_{n \rightarrow \omega} \|x_n\|_{2, \tau_n} = 0 \}$$

is a quotient of a WEP C^* -algebra
 QWEP.

Lemma If $A \subset B$ rwi then

$$B \text{ (Q)WEP} \Rightarrow A \text{ (Q)WEP}.$$

PF \Downarrow

For QWEP we need some facts about the double dual
 but taking these for granted:

Let C be a C^* alg. w/ WEP and $\pi: C \rightarrow B$, $J := \ker \pi$

Let $\psi: B^{**} \rightarrow A^{**}$ the normal cond. exp from Lem (1')

$$\text{Then } \pi^{-1}(A)^{**} \cong J^{**} \oplus A^{**} \subset J^{**} \oplus A^{**} \cong C^{**}$$

$$\leadsto \text{id}_{J^{**}} \oplus \psi: C^{**} \rightarrow \pi^{-1}(A)^{**} \text{ is a cond exp.}$$

$$\Rightarrow \pi^{-1}(A) \overset{\text{rwi}}{\hookrightarrow} C \Rightarrow \pi^{-1}(A) \text{ has WEP } \square$$

| | |
|--|---|
| $\begin{aligned} & A \subset B \text{ rwi} \\ & \Rightarrow A \otimes_{\max} C \subset B \otimes_{\max} C \\ & \quad \neq C \end{aligned}$ | $\begin{aligned} & B \text{ WEP} \\ & + A \subset B \text{ rwi} \end{aligned} \Rightarrow \begin{aligned} & C^*(F) \otimes_{\max} B = C^*(F) \otimes_{\min} B \\ & \quad \cup \\ & C^*(F) \otimes_{\max} A \quad C^*(F) \otimes_{\min} B \end{aligned}$ |
|--|---|

3) Fact If (M, τ) is a tracial vN alg, then any vN sub-alg $N \subset M$ is the range of a conditional expectation $M \rightarrow N$

Upshot Since $(\mathbb{R}^{\omega}, \text{tr}^{\omega})$ is a tracial vN alg, every von Neumann subalgebra of \mathbb{R}^{ω} inherits QWEP

Now, let's see why CEP \Rightarrow QWEP:

Prop (Kirchberg), If $\{A_i\}$ is an increasing net of QWEP C^* -algebras in $B(\mathcal{H})$, then $(\cup_i A_i)''$ is QWEP

Defn: A vN alg is semifinite if it can be written as $(\cup_i M_i)''$ where each M_i is a tracial vN alg with a separable predual.

Hence CEP \Rightarrow All semifinite vN algebras are QWEP

Corollary (to Tomita-Takesaki Modular Theory)

For any vN alg M , \exists an action $\mathbb{R} \curvearrowright M$ (called the modular action) s.t. $M \rtimes_{\alpha} \mathbb{R}$ is a semifinite vN alg. Moreover \exists a conditional expectation $M \rtimes_{\alpha} \mathbb{R} \rightarrow M$ (which is the pt attract limit of normal vcp maps $M \rtimes_{\alpha} \mathbb{R} \rightarrow M$).

Hence CEP \Rightarrow all vN algebras are QWEP

Prop (Kirchberg) For any C^* -alg A
 A is QWEP $\Leftrightarrow A^{**}$ is QWEP.

Hence CEP \Rightarrow all C^* -algebras are QWEP

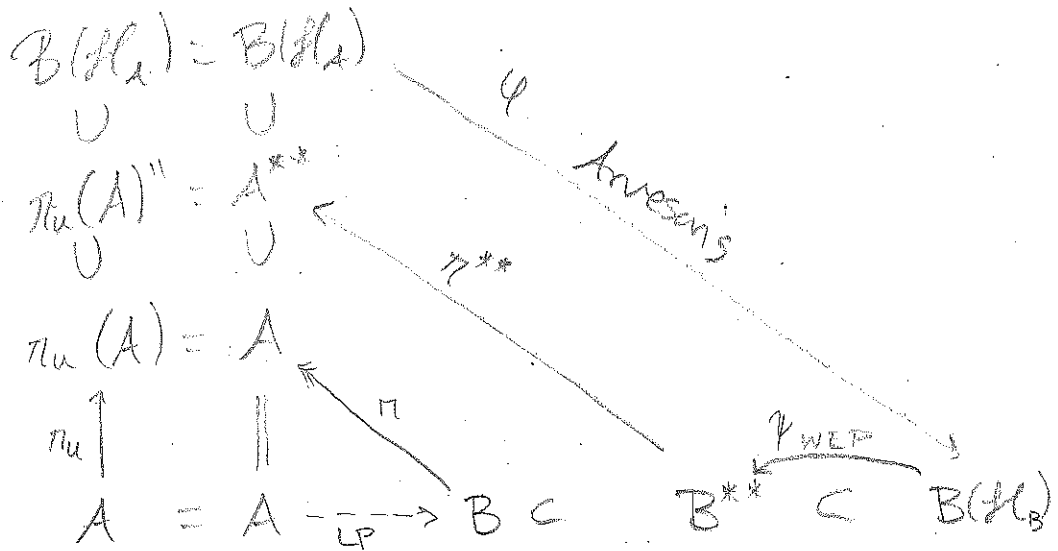
Permanence Properties of (Q)WEP

- 1) If $\{A_i\}$ is a family of (Q)WEP C^* -algebras, then $\prod_{i \in I} A_i$ is (Q)WEP
- 2) If $A \xrightarrow{\text{quot}} B$ and B is (Q)WEP, then so is A .
- 3) If $\{A_i\}$ is an inc. net of QWEP C^* -algebras in $B(\mathcal{H})$, then $\overline{\bigcup A_i}$ and $(\bigcup A_i)''$ is QWEP.
- 4) A is QWEP iff A^{**} is QWEP
- 5) $A \vee N$ alg M is QWEP iff M is.

Next, we show (2) \Rightarrow (7)

(6)

The claim amounts to showing $LLP + QWEP \Rightarrow WEP$
 We are going to show it for LP and not worry w/ taking cluster pts
 Suppose A is a C^* -alg w/ LP, B a C^* -alg
 with WEP, and $\pi: B \rightarrow A$.



Then $LP \Rightarrow \exists$ ucp map $A \rightarrow B \subset B(H_B)$
 \cap
 $B(H_A)$

Arveson's extends this to a ucp map $B(H_A) \xrightarrow{\psi} B(H_B)$

B WEP $\Rightarrow \exists$ weak cond exp. $B(H_B) \xrightarrow{\psi} B^{**}$
 extending $B \subset B^{**}$

Let $\pi^{**}: B^{**} \rightarrow A^{**}$ be the normal extension of

$\pi: B \rightarrow A$. Then $\pi^{**} \circ \psi \circ \varphi: B(H_A) \rightarrow A^{**}$

is our desired map. \square

Since we have already seen (7) \Rightarrow (3),
 let's move on to (3) \Rightarrow (1)

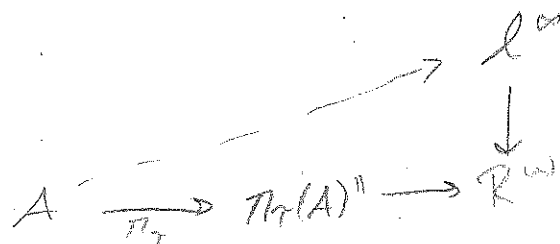
Amenable Traces

(7)

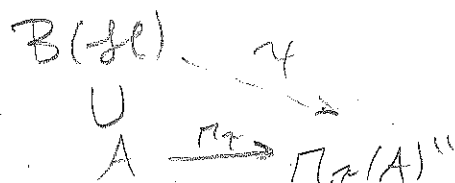
Thm/Definition (Connes, Kirchberg)

Let A be a separable unital C^* -alg and $\tau \in T(A)$ a tracial state

- 1) \exists an embedding $\pi_\tau(A)'' \subset \mathbb{R}^\omega$ s.t.
 $\pi_\tau: A \rightarrow \pi_\tau(A)'' \subset \mathbb{R}^\omega$ has a ucp lift
 $A \rightarrow \ell^\infty(\mathbb{R})$ and $\text{tr}^\omega \circ \pi_\tau = \tau$



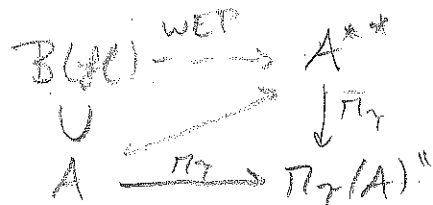
- 2) For any ^{nondeg} embedding $A \subset B(\mathcal{H})$, \exists a ucp map
 $\psi: B(\mathcal{H}) \rightarrow \pi_\tau(A)''$ s.t. $\psi(a) = \pi_\tau(a) \forall a \in A$



We call a trace satisfying either of these conditions amenable.

Amenable Traces and WEP.

Suppose $A \subset B(\mathcal{H})$ has WEP and a tracial state τ



Then τ is amenable.

Hence if $C^*(\mathbb{F})$ has WEP, then every trace on $C^*(\mathbb{F})$ is amenable.

(3) \Rightarrow (1) Outline

(8)

Suppose M is a II_1 -factor with separable predual M_* and faithful tracial state τ .

Since M_* is separable, M contains a countable family of unitaries whose span is dense in the wk^* -topology on M .

From this we get a $*$ -hom $\psi: C^*(\mathbb{F}_\infty) \rightarrow M$ with wk^* -dense image.

Then $\tau \circ \psi$ is a tracial state on $C^*(\mathbb{F}_\infty)$ and we can identify $M \cong \pi_{\tau \circ \psi}(C^*(\mathbb{F}_\infty))''$ (uniqueness of GNS)

Now if $C^*(\mathbb{F}_\infty)$ has WEP, then $\tau \circ \psi$ is amenable, which gives us the tracial embedding

$$M \cong \pi_{\tau \circ \psi}(C^*(\mathbb{F}_\infty))'' \subset R^w$$

□

Now, how about those \mathbb{Z}_m^{*k} .

Prop Let C be a C^* -alg w/ LLP st. \exists a copy $C^*(\mathbb{F}_2) \subset C$ w/i.

Then for any C^* -alg B , B has WEP iff $C \otimes_{\max} B = C \otimes_{\min} B$. (ie C characterizes WEP)

Proof If B has WEP then by Kirchberg's thm

$$C \otimes_{\max} B = C \otimes_{\min} B \text{ since } C \text{ has LLP.}$$

OTOH if " ", then $C^*(\mathbb{F}_2) \otimes_{\max} B \subset C \otimes_{\max} B$

$$\parallel \\ C^*(\mathbb{F}_2) \otimes_{\min} B \subset C \otimes_{\min} B$$

Thm (Boca/Pisier)

LLP is preserved under taking free products.

9) Since each $C^*(\mathbb{Z}/m\mathbb{Z})$ is s.d., it has the LLP.
 Hence for any $k > 2$, $C^*(\mathbb{Z}_m^{*k})$ has LLP too.
 When either $k > 2$ or $m > 2$, we can find
 a copy of \mathbb{F}_2 inside $\mathbb{Z}_m^{*k} \rightsquigarrow C^*(\mathbb{F}_2) \xrightarrow{\text{w.i.}} C^*(\mathbb{Z}_m^{*k})$

Note that if any C^* -alg that characterizes WEP has WEP, then every C^* -alg that characterizes WEP has WEP.

Indeed, suppose C and C' characterize WEP.
 Then both have LLP since they must have
 a ! tensor norm w/ WEP C^* -alg $\mathbb{B}(\mathbb{H})$.

Hence if C has WEP, then $C \otimes_{\max} C' = C \otimes_{\min} C'$
 since C' has LLP, but this implies C'
 has WEP since C characterizes WEP.

Hence we get (4) \Leftrightarrow (6).

Why (6)?

This the OA version of Tsirelson's Problem.

$C^*(\mathbb{Z}_m^{*k})$ is generated by k unitaries, each
 with $|o(u)| = m$. $C^*(\mathbb{Z}_m^{*k}) = C_u^*(\{a_x : a_x^m = 1, 1 \leq x \leq k\})$
 $a_x a_x^* = 1 = a_x^* a_x$

\rightsquigarrow It is also generated by their spectral
 projections, \rightsquigarrow Projection valued measures.

$\rightsquigarrow C^*(\mathbb{Z}_m^{*k}) = C_u^*(\{P_{x,a} : P_{x,a}^2 = P_{x,a}^* = P_{x,a}, \sum_{a=1}^m P_{x,a} = 1, 1 \leq x \leq k\})$
 Projection valued measures

Tsirelson is asking about all the ways such
 projection valued measures can commute.