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Question (Connes 76). (Popa)

Does every separably-acting ~~II_1-factor~~ tracial vN alg embed into some ultrapower R^ω of the hyperfinite II_1 factor R?

Theorem (Kirchberg) The following conjectures are equiv

1) CEP
2) Every C^*-algebra is QWEP (Kirchberg's QWEP conj)
3) $C^*(\mathbb{F})$ has WEP for any (every) non-abelian free gp
4) $C^*(\mathbb{F}) \otimes_{max} C^*(\mathbb{F}) = C^*(\mathbb{F}) \otimes_{min} C^*(\mathbb{F})$ " " " " " "
6) $C^*(\mathbb{Z}_m^{*k}) \otimes_{max} C^*(\mathbb{Z}_m^{*k}) = C^*(\mathbb{Z}_m^{*k}) \otimes_{min} C^*(\mathbb{Z}_m^{*k})$
For each $k, m \geq 2$ with either $k>2$ or $m>2$
$(\mathbb{Z}_m^{*k} = \ast_{j=1}^{k} \mathbb{Z}/m\mathbb{Z})$
5) $C^*(\mathbb{F} \times \mathbb{F})$ is RFD " " " " "
7) LLP $\Rightarrow$ WEP

Let's recap what we know:

A C^* alg A has WEP iff $A \otimes_{max} C^*(\mathbb{F}) = A \otimes_{min} C^*(\mathbb{F})$ for any (all) non-ab free gp(s) $\mathbb{F}$.

Hence $(3) \Leftrightarrow (4)$.

By comparing universal properties, one can see that $C^*(\mathbb{F}) \otimes_{max} C^*(\mathbb{F}) = C^*(\mathbb{F} \times \mathbb{F})$.

Moreover, the minimum tensor product of any two RFD C^* algebras A, B is RFD
(Indeed, take the tensor products of respective separating families of f.dim'l reps.)

Since $C^*(\mathbb{F})$ is RFD, $(4) \Rightarrow (5)$ follows.

In fact, given two RFD C^*-algebras A, B, $A \otimes_{max} B$ is RFD iff $A \otimes_{min} B = A \otimes_{max} B$.
(Essentially b/c any rep $\pi: A \odot B \to M_n$ extends to $A \otimes_{min} B$ too)

So, we know $(3) \Leftrightarrow (4) \Leftrightarrow (5)$.

Since $A \otimes_{max} B$ & $A \otimes_{min} B$ have sep families of f.d. reps,

want to show that for any f.d. rep $\pi : A \otimes_{max} B \to M_n$

$\exists$ rep $\pi' : A \otimes_{min} B \to M_n$ s.t. $\|\pi'(x)\| \geq \|\pi(x)\|$ $\forall x \in A \odot B$.

<u>Claim</u> Any f.d rep $\pi : A \odot B \to M_n$ extends to a rep of $A \otimes_{min} B$.

Let π_A & π_B be the restrictions. ($\pi = \pi_A \times \pi_B$) $(\pi(a \otimes b) = \pi_A(a)\pi_B(b)$ $\forall a \in A, b \in B)$

Then the natural inclusions $\iota_A : \pi_A(A) \hookrightarrow M_n$, $\iota_B : \pi_B(B) \hookrightarrow M_n$

induce $\iota_A \times \iota_B : \pi_A(A) \otimes_{max} \pi_B(B) \to M_n$

w/ $\iota_A \times \iota_B(\pi_A(a) \otimes \pi_B(b)) = \pi_A(a)\pi_B(b)$ $\forall a \in A, b \in B$

OTOH, we have a map $\pi_A \otimes \pi_B : A \otimes_{min} B \to \pi_A(A) \otimes_{min} \pi_B(B)$

Since $\pi_A(A) \otimes_{max} \pi_B(B) = \pi_A(A) \otimes_{min} \pi_B(B)$

we can define $\tilde{\pi} : A \otimes_{min} B \to M_n$ by $\tilde{\pi} := (\iota_A \times \iota_B) \circ (\pi_A \otimes \pi_B)$

$\rightsquigarrow \tilde{\pi}(a \otimes b) = (\iota_A \times \iota_B)(\pi_A(a) \otimes \pi_B(b)) = \pi_A(a)\pi_B(b) = \pi(a \otimes b)$

Let's talk about (1) $\Rightarrow$ (2) $\Rightarrow$ (7) $\Rightarrow$ (3) $\Rightarrow$ (1)
<u>Ok.</u> (7) $\Rightarrow$ (3) we know

<u>Recall</u> For C^*-algebras $A \subset B$, a map $\Psi : B \to A^{**}$ extending the canonical embedding $A \subset A^{**}$ is a <u>weak conditional expectation</u>.

When such a Ψ exists, we say $A \hookrightarrow B$ rwi.
When every embedding of A (or equiv any $A \hookrightarrow \mathrm{Inj}$) is rwi, then A has WEP $\Longleftrightarrow$

$$\begin{tikzcd} & \mathrm{Inj} \arrow[dr] & \\ B \supset A \arrow[ur, dashed] \arrow[rr] & & A^{**} \end{tikzcd}$$

<u>Examples</u> All injective C^*-algebras have WEP including all semidiscrete (& hence also all hyperfinite) vN alg

$$\begin{tikzcd} B \supset A \arrow[r, "\rho"] \arrow[drr, "\Psi_n \circ \rho", "\text{(Arveson)}"'] & \mathcal{M} \arrow[r] \arrow[dr, "\psi_n"] & \mathcal{M} \\ & & M_n \arrow[u, "\varphi_n"'] \end{tikzcd}$$

$\varphi_n \circ \psi_n \longrightarrow \mathrm{id}_{\mathcal{M}}$ pt. ultrawkly
$(\eta(\varphi_n \circ \psi_n(a)) \to \eta(a)\ \forall \eta \in \mathcal{M}_*)$

So $\varphi_n \circ \widetilde{\psi_n \circ \rho} : B \to \mathcal{M}$ ucp, $\varphi_n \circ \widetilde{\psi_n \circ \rho}|_A = \varphi_n \circ \psi_n \circ \rho$
and $\varphi_n \circ \psi_n \circ \rho \longrightarrow \rho$ pt ultrawkly

Then $\varphi_n \circ \widetilde{\psi_n \circ \rho}$ has a clusterpt in the ultra-wk top
(by Banach-Alaoglu)

In particular, $\mathcal{R}$ has WEP.

<u>Prop (Kir)</u> WEP is closed under taking direct prod.

398 <u>Proof</u> Let $\{A_i\}_i$ be a family of WEP C^*-alg.s w/ $A_i \subset B(\mathcal{H}_i)$. Then $\prod_i B(\mathcal{H}_i)$ is injective and so, it suffices to find $\varphi : \prod_i B(\mathcal{H}_i) \to (\prod_i A_i)^{**}$ extending $\prod_i A_i \hookrightarrow (\prod_i A_i)^{**}$.

<u>Claim</u> For any f.d. subsp $E \subset \prod_i B(\mathcal{H}_i)$ & $\varepsilon > 0$, $\exists$ a contractive linear map $\psi : E \to \prod_i A_i =: A$ s.t. $\| \psi|_{E \cap A} - \mathrm{id}_{E \cap A} \| < \varepsilon$.

3) <u>Principle of Local Reflexivity</u>

Let X be a Banach sp & $E \subset X^{**}$ f.dim'l subsp. Then $\exists$ a net $\varphi_i : E \to X$ of contractions converging to id_E ptwk* s.t. $\{\varphi_i|_{E\cap X}\}_i$ converges to $id_{E\cap X}$ pt-norm

<u>Lemma</u> Let $A \subset B$ be C^*-alg. TFAE

(1) $A \hookrightarrow B$ rwi (1') $\exists$ Normal cond exp $B^{**} \to A^{**}$

(2) For any f.d. $E \subset B$ & $\varepsilon > 0$, $\exists$ contractive linear map $\psi : E \to A$ s.t. $\|\psi|_{E\cap A} - id_{E\cap A}\| < \varepsilon$.

<u>Pf</u>

(1) $\Rightarrow$ (2)

$$\begin{tikzcd} E \subset B \arrow[dr, "\varphi"] & \\ A \arrow[u, phantom, "\cup"] \arrow[r, hook, "\iota"] & A^{**} \supset \varphi(E) \arrow[l, bend left, "\text{Princp. Loc. Reflex}", "\varphi_i"'] \end{tikzcd}$$

set $\psi_i = \varphi_i \circ \varphi$. Since $\varphi|_A = \iota$,

$$\psi_i|_{E\cap A} = \varphi_i|_{E\cap A} = \varphi_i|_{\varphi(E)\cap A} \xrightarrow[\|\cdot\|]{\text{ptwse}} id_{\varphi(E)\cap A} = id_{E\cap A}$$

a compactness argument (for (E1)) gives the uniformity

(2) $\Rightarrow$ (1) For each $E \subset B$ f.dim & $\varepsilon > 0$, let $\psi_{E,\varepsilon} : E \to A$ contraction w/ $\|\psi|_{E\cap A} - id_{E\cap A}\| < \varepsilon$.
Let $\varphi : B \to A^{**}$ be a wk*-cluster pt.
$\rightsquigarrow$ φ is a contractive linear map w/ $\varphi|_A = id_A$.
Then φ extends to a wk*-cts contractive linear map $\varphi^{**} : B^{**} \to A^{**}$ (on the level of Banach spaces)
Then φ^{**} is a contraction s.t. $\varphi|_{A^{**}} = id_{A^{**}}$.
By Tomiyama's Theorem, φ^{**} is a conditional expectation & hence cpc.

By the Lemma, we know that for each $E_i \subset B(\mathcal{H}_i) \cup A_i$
$\exists$ a contractive linear map $\psi_i : \Sigma_i \to A_i$ s.t.
$\|\psi_i|_{E_i \cap A_i} - \mathrm{id}|_{E_i \cap A_i}\| < \varepsilon$. (4)
Then $\psi := \oplus \psi_i : \Sigma \to \prod A_i$ is contractive w/
$\|\psi|_{E \cap \prod A_i} - \mathrm{id}_{E \cap \prod A_i}\| < \varepsilon$, and we are done
by the lemma. $\square$

<u>Cor</u> $\ell^\infty(R) = \prod_{\mathbb{N}} R$ has WEP

Then for any (free) ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$,

$$R^\omega = \ell^\infty(R) \Big/ \{(x_n) : \lim_{n \to \omega} \|x_n\|_{2,\tau_n} = 0\}$$

is a <u>quotient of a WEP C^*-algebra</u>
QWEP.

<u>Lemma</u> If $A \subset B$ rwi then
B (Q)WEP $\Rightarrow$ A (Q)WEP.

<u>Pf</u>
For QWEP we need some facts about the double dual but taking these for granted:
Let C be a C^*-alg w/ WEP and $\pi : C \twoheadrightarrow B$, $J := \ker \pi$
Let $\psi : B^{**} \to A^{**}$ the normal cond. exp from Lem (7')
Then $\pi^{-1}(A)^{**} \cong J^{**} \oplus A^{**} \subset J^{**} \oplus A^{**} \cong C^{**}$.
$\rightsquigarrow \mathrm{id}_{J^{**}} \oplus \psi : C^{**} \to \pi^{-1}(A)^{**}$ is a cond exp.
$\Rightarrow \pi^{-1}(A) \overset{rwi}{\hookrightarrow} C \Rightarrow \pi^{-1}(A)$ has WEP $\square$

$A \subset B$ rwi
$\Rightarrow A \otimes_{max} C \subset B \otimes_{max} C$
$\forall C$

B WEP
$+ A \subset B$ rwi $\Rightarrow$ $C^*(\mathbb{F}) \otimes_{max} B = C^*(\mathbb{F}) \otimes_{min} B$

$$\begin{array}{ccc} C^*(\mathbb{F}) \otimes_{max} B & = & C^*(\mathbb{F}) \otimes_{min} B \\ \cup & & \cup \\ C^*(\mathbb{F}) \otimes_{max} A & & C^*(\mathbb{F}) \otimes_{min} B \end{array}$$

5)

<u>Fact</u> If (M, τ) is a tracial vN alg, then any vN sub-alg $N \subset M$ is the range of a conditional expectation $M \to N$

<u>Upshot</u> Since (R^ω, tr^ω) is a tracial vN alg, every von Neumann subalgebra of R^ω inherits QWEP

Now, let's see why CEP $\Rightarrow$ QWEP:

<u>Prop (Kirchberg)</u> If $\{A_i\}$ is an increasing net of QWEP C^*-algebras in $B(\mathcal{H})$, then $(\bigcup_i A_i)''$ is QWEP

<u>Defn</u> A vN alg is <u>semifinite</u> if it can be written as $(\bigcup_i M_i)''$ where each M_i is a tracial vN alg with a separable predual.

Hence CEP $\Rightarrow$ All semifinite vN algebras are QWEP

<u>Corollary (to Tomita-Takesaki Modular Theory)</u>
For any vN alg M, $\exists$ an action $\mathbb{R} \overset{\alpha}{\curvearrowright} M$ (called the modular action) s.t. $M \rtimes_\alpha \mathbb{R}$ is a semi-finite vN alg. Moreover $\exists$ a conditional expectation $M \rtimes_\alpha \mathbb{R} \to M$ (which is the pt ultraweak limit of normal ucp maps $M \rtimes_\alpha \mathbb{R} \to M$).

Hence CEP $\Rightarrow$ all vN algebras are QWEP

<u>Prop (Kirchberg)</u> For any C^*-alg A
A is QWEP $\Leftrightarrow$ A^{**} is QWEP.

Hence CEP $\Rightarrow$ all C^*-algebras are QWEP

Permanence Properties of (Q) WEP

1) If $\{A_i\}_i$ is a family of (Q)WEP C^*-algebras, then $\prod_{i \in I} A_i$ is (Q)WEP
2) If $A \subset_{rwi} B$ and B is (Q)WEP, then so is A.
3) If $\{A_i\}_i$ is an inc. net of QWEP C^*-algebras in $B(\mathcal{H})$, then $\overline{\cup A_i}$ and $(\cup A_i)''$ is QWEP.
4) A is QWEP iff A^{**} is QWEP
5) A vN alg $\mathcal{M}$ is QWEP iff $\mathcal{M}'$ is.

Next, we show (2) ⇒ (7)

The claim amounts to showing LLP + QWEP ⇒ WEP

We are going to show it for LP and not worry w/ taking cluster pts

Suppose A is a C^*-alg w/ LP, B a C^*-alg with WEP, and $\pi: B \twoheadrightarrow A$.

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H}_A) \arrow[r, phantom, "="] & \mathcal{B}(\mathcal{H}_A) \arrow[dddrrr, dashed, "\varphi", "\text{Arveson's}"'] & & & \\
\pi_u(A)'' \arrow[u, phantom, sloped, "\subset"] \arrow[r, phantom, "="] & A^{**} \arrow[u, phantom, sloped, "\subset"] & & & \\
\pi_u(A) \arrow[u, phantom, sloped, "\subset"] \arrow[r, phantom, "="] & A \arrow[u, phantom, sloped, "\subset"] & & & \\
A \arrow[u, "\pi_u"] \arrow[r, phantom, "="] & A \arrow[u, equal] \arrow[r, dashed, "LP"'] & B \arrow[ul, "\pi"'] \arrow[r, phantom, "\subset"] & B^{**} \arrow[uull, "\pi^{**}"'] \arrow[r, phantom, "\subset"] & \mathcal{B}(\mathcal{H}_B) \arrow[l, bend right, "\psi_{WEP}"']
\end{tikzcd}$$

Then LP ⇒ ∃ ucp map $\underset{\mathcal{B}(\mathcal{H}_A)}{\overset{A}{\cap}} \longrightarrow B \subset \mathcal{B}(\mathcal{H}_B)$

Arveson's extends this to a ucp map $\mathcal{B}(\mathcal{H}_A) \xrightarrow{\varphi} \mathcal{B}(\mathcal{H}_B)$

B WEP ⇒ ∃ weak cond exp. $\mathcal{B}(\mathcal{H}_B) \xrightarrow{\psi} B^{**}$ extending $B \subset B^{**}$

Let $\pi^{**}: B^{**} \longrightarrow A^{**}$ be the normal extension of $\pi: B \to A$. Then $\pi^{**} \circ \psi \circ \varphi : \mathcal{B}(\mathcal{H}_A) \to A^{**}$ is our desired map. □

Since we have already seen (7) ⇒ (3), let's move on to (3) ⇒ (1)

Amenable Traces

Thm/Definition (Connes, Kirchberg)

Let A be a separable unital C^*-alg and $\tau \in T(A)$ a tracial state

1) $\exists$ an embedding $\pi_\tau(A)'' \subset R^\omega$ s.t. $\pi_\tau : A \to \pi_\tau(A)'' \subset R^\omega$ has a ucp lift $A \to \ell^\infty(R)$ and $tr^\omega \circ \pi_\tau = \tau$

$$\begin{tikzcd} & & \ell^\infty \arrow[d] \\ A \arrow[r, "\pi_\tau"'] \arrow[urr, dashed] & \pi_\tau(A)'' \arrow[r] & R^\omega \end{tikzcd}$$

2) For any nondeg embedding $A \subset B(\mathcal{H})$, $\exists$ a ucp map $\psi : B(\mathcal{H}) \to \pi_\tau(A)''$ s.t. $\psi(a) = \pi_\tau(a)$ $\forall a \in A$

$$\begin{tikzcd} B(\mathcal{H}) \arrow[dr, dashed, "\psi"] & \\ A \arrow[u, phantom, "\cup"] \arrow[r, "\pi_\tau"] & \pi_\tau(A)'' \end{tikzcd}$$

We call a trace satisfying either of these conditions <u>amenable</u>.

Amenable Traces and WEP.

Suppose $A \subset B(\mathcal{H})$ has WEP and a tracial state τ.

$$\begin{tikzcd} B(\mathcal{H}) \arrow[r, dashed, "WEP"] & A^{**} \arrow[d, "\pi_\tau"] \\ A \arrow[u, phantom, "\cup"] \arrow[r, "\pi_\tau"] \arrow[ur, dashed, leftarrow] & \pi_\tau(A)'' \end{tikzcd}$$

Then τ is amenable.

Hence if $C^*(\mathbb{F})$ has WEP, then every trace on $C^*(\mathbb{F})$ is amenable.

(3) ⇒ (1) Outline

Suppose $\mathcal{M}$ is a II_1-factor with separable predual $\mathcal{M}_*$ and faithful tracial state τ.
Since $\mathcal{M}_*$ is separable, $\mathcal{M}$ contains a countable family of unitaries whose span is dense in the wk*-topology on $\mathcal{M}$.
From this we get a $*$-hom $\Psi : C^*(\mathbb{F}_\infty) \to \mathcal{M}$ with wk*-dense image.
Then $\tau \circ \Psi$ is a tracial state on $C^*(\mathbb{F}_\infty)$ and we can identify $\mathcal{M} \cong \pi_{\tau\circ\Psi}(C^*(\mathbb{F}_\infty))''$ (uniqueness of GNS)

Now if $C^*(\mathbb{F}_\infty)$ has WEP, then $\tau\circ\Psi$ is amenable, which gives us the tracial embedding

$$\mathcal{M} \cong \pi_{\tau\circ\Psi}(C^*(\mathbb{F}_\infty))'' \subset R^\omega$$

$\square$

Now, how about those $\mathbb{Z}_m^{*k}$?

<u>Prop</u> Let C be a C^*-alg w/ LLP s.t. $\exists$ a copy $C^*(\mathbb{F}_2) \subset C$ rwi.
Then for any C^*-alg B, B has WEP iff $C \otimes_{\max} B = C \otimes_{\min} B$. (i.e. C <u>characterizes WEP</u>)

<u>Proof</u> If B has WEP then by Kirchberg's thm $C \otimes_{\max} B = C \otimes_{\min} B$ since C has LLP.
OTOH if $\underbrace{\quad ''\quad}$, then $C^*(\mathbb{F}_2) \otimes_{\max} B \subset C \otimes_{\max} B$

$$C^*(\mathbb{F}_2) \otimes_{\min} B \qquad \overset{\Vert}{C \otimes_{\min} B}$$

<u>Thm (Boca/Pisier)</u>
LLP is preserved under taking free products.

9) Since each $C^*(\mathbb{Z}/m\mathbb{Z})$ is f.d., it has the LLP. Hence for any $k>2$, $C^*(\mathbb{Z}_m^{*k})$ has LLP too. When either $k>2$ or $m>2$, we can find a copy of $\mathbb{F}_2$ inside $\mathbb{Z}_m^{*k} \rightsquigarrow C^*(\mathbb{F}_2) \xrightarrow{\text{rwi}} C^*(\mathbb{Z}_m^{*k})$

Note that if <u>any</u> C^*-alg that characterizes WEP <u>has</u> WEP, then <u>every</u> C^*-alg that characterizes WEP <u>has</u> WEP.

Indeed, suppose C and C' characterize WEP. Then both have LLP since they must have a ! tensor norm w/ WEP C^*-alg $\mathcal{B}(\mathcal{H})$.

Hence if C has WEP, then $C \otimes_{max} C' = C \otimes_{min} C'$ since C' has LLP, but this implies C' has WEP since C characterizes WEP.

Hence we get $(4) \Leftrightarrow (6)$.

Why (6)?

This the OA version of Tsirleson's Problem.

$C^*(\mathbb{Z}_m^{*k})$ is generated by k unitaries, each with $|\sigma(u)| = m$. $C^*(\mathbb{Z}_m^{*k}) = C_u^*(\{a_x : a_x^m = 1, \ a_x a_x^* = 1 = a_x^* a_x, \ 1 \le x \le k\})$

$\rightsquigarrow$ It is also generated by their spectral projections $\rightsquigarrow$ Projection valued measures

$\rightsquigarrow C^*(\mathbb{Z}_m^{*k}) = C_u^*(\{P_{x,a} : \underbrace{P_{x,a}^2 = P_{x,a}^* = P_{x,a}, \ \sum_{a=1}^m P_{x,a} = 1}_{\text{projection valued measures}}, \ 1 \le x \le k\})$

Tsirleson is asking about all the ways such projection valued measures can commute.