Residually Finite Stuffs

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1 Non-exact C*-algebra

We pick-up from where we left off in talk 3 about exact C*-algebras. We learnt what an exact C*-algebra is and how exact sequences of C*-algebras work. In the first section of this talk, we will see an example of a C*-algebra which is not exact. For that we will need some preliminary facts. Through proving the non-exactness of a C*-algebra, we will also witness how crucial passing to tensor products can be even when it seems like it is not required. Note that through out this talk we will mean "faithful tracial state" when we say "trace".

Recall that a group Γ is residually finite if there exist finite-index normal subgroups $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ with trivial intersection.

Definition 1.1. (Left regular representation) The unitary representation $\lambda : \Gamma \to B(\ell^2(\Gamma))$ given by

$$(\lambda(s)f)_t = f_{s^{-1}t}$$

Similarly for the right regular representation ρ.

Lemma 1.2. *If Γ is residually finite, then there exists a state μ on $C^*(\Gamma) \otimes C^*(\Gamma)$ such that for finite sums*

$$x = \sum_{s\in\Gamma} \alpha_s s \ , \ y = \sum_{t\in\Gamma} \beta_t t$$

in $C^*(\Gamma)$, we have

$$\mu(x \otimes y) = \sum_{s\in\Gamma} \alpha_s \beta_s$$

Proof. Γ is residually finite
$\implies$ each Γ/Γ_n is a finite group.
$\implies$ There exists a sequences of states μ_n on $C^*(\Gamma/\Gamma_n) \otimes C^*(\Gamma/\Gamma_n)$ satisfying the above formula for elements in Γ/Γ_n.
We also have quotient mappings $\pi_n : C^*(\Gamma) \to C^*(\Gamma/\Gamma_n)$.
$\implies$ There exist tensor product $*$-homomorphisms

$$\pi_n \otimes \pi_n : C^*(\Gamma) \otimes C^*(\Gamma) \to C^*(\Gamma/\Gamma_n) \otimes C^*(\Gamma/\Gamma_n).$$

Since the intersection of the Γ_n's is the identity element, any cluster point of the states $\mu_n \circ (\pi_n \otimes \pi_n)$ must satisfy the above formula. $\square$

Proposition 1.3. *If* Γ *is residually finite, then the product map*

$$\lambda \times \rho : C^*(\Gamma) \odot C^*(\Gamma) \to B(\ell^2(\Gamma))$$

is continuous w.r.t. the spatial (min) tensor product norm.

Proof. We prove this by simply showing that the image under $\lambda \times \rho$ sits inside $C^*(\Gamma) \otimes C^*(\Gamma)$ as C*-subalgebra.
To this end, let $\pi : C^*(\Gamma) \otimes C^*(\Gamma) \to B(H)$ be the GNS representation of the state constructed in Lemma 1.2.
Since GNS representation is unique, the following two representations are unitarily equivalent.

$$\pi|_{C^*(\Gamma)\odot C^*(\Gamma)} : C^*(\Gamma) \odot C^*(\Gamma) \to B(H)$$
$$\lambda \times \rho : C^*(\Gamma) \odot C^*(\Gamma) \to B(\ell^2(\Gamma))$$

This is because $\delta_e \in \ell^2(\Gamma)$ is a cyclic vector for the algebra $\lambda \times \rho(C^*(\Gamma) \odot C^*(\Gamma))$ whose corresponding vector functional agrees with μ.
So the C*-algebra generated by $(\lambda \times \rho)(C^*(\Gamma) \odot C^*(\Gamma))$ is a quotient of $C^*(\Gamma) \otimes C^*(\Gamma)$. $\square$

Before stating the important theorem, we invoke a technical result the proof of which can be seen in [2] §2.6.

Theorem 1.4. *For a locally compact group* Γ*, the following are equivalent ([1], [6]).*

i. Γ *is amenable*

ii. $C_r^*(\Gamma) = C^*(\Gamma)$

iii. $C^*(\Gamma)$ *is nuclear*

We know from talks 3 and 4 that nuclearity implies exactness and that the converse is false.

Theorem 1.5. *Let* Γ *be a residually finite discrete group. Then the following are equivalent.*

i. Γ *is amenable*

ii. $C^*(\Gamma)$ *is exact*

iii. Taking J to be the kernel of the quotient map $C^*(\Gamma) \to C_r^*(\Gamma)$, the following sequence is exact

$$0 \to J \otimes C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma) \to C_r^*(\Gamma) \otimes C^*(\Gamma) \to 0$$

Proof. (i) $\implies$ (ii) since amenability implies nuclearity (Theorem 1.4) which in turn implies exactness.
(ii) $\implies$ (iii) has been proved in talk 3.
(iii) $\implies$ (i)
The map in Proposition 1.3 extends to a $*$-homomorphism

$$\lambda \otimes \rho : C^*(\Gamma) \otimes_{min} C^*(\Gamma) \to B(\ell^2(\Gamma))$$

Since $J \otimes C^*(\Gamma)$ belongs to the kernel of the above map, the $*$-homomorphism factors through

$$\frac{C^*(\Gamma \otimes C^*(\Gamma)}{J \otimes C^*(\Gamma)} \cong C_r^*(\Gamma) \otimes C^*(\Gamma).$$

Then Proposition 1.3 implies that we have a $*$-homomorphism $\pi : C_r^*(\Gamma) \otimes C^*(\Gamma) \to B(\ell^2(\Gamma))$ such that

$$\pi(x \otimes y) = x\rho(y)$$

for all $x \in C_r^*(\Gamma)$ and $y \in C^*(\Gamma)$.
Now we use "The Trick"[1] as stated in talk 2 and used in talk 5.
Recall that the commutant of the range of right regular representation is $L(\Gamma)$.
So we get a u.c.p map $\phi : B(\ell^2(\Gamma)) \to L(\Gamma)$ such that it does not change elements in $C_r^*(\Gamma)$.
Let τ be the canonical vector trace $\tau(x) = \langle x\delta_e, \delta_e \rangle$ on $L(\Gamma)$.
Consider the state $\eta = \tau \circ \phi$ on $B(\ell^2(\Gamma))$.
On restricting this to $\ell^\infty(\Gamma) \subset B(\ell^2(\Gamma))$ we get an invariant mean. □

We finish this section by stating that the required example of a non-exact C*-algebra is the group C*-algebra of a residually finite group which is not amenable.

Corollary 1.6. *If Γ is a non-amenable residually finite group then $C^*(\Gamma)$ is not exact*

What is the most popular group which is not amenable? The free group of course.

[1]The trick: Let $A \subset B$ and C s.t. $A \otimes_{min} C \subset B \otimes_{min} C$. Given representations π_A and π_C, with commuting ranges, there exists a ccp map $\phi : B \to \pi_C(C)'$ which extends π_A.

2 $C^*(\mathbb{F}_2)$ is RFD

The free group on two generators $\mathbb{F}_2$ is residually finite. This means that we can, in some sense, approximate $\mathbb{F}_2$ by finite groups. Among the several interesting properties that $\mathbb{F}_2$ has, recall that it contains every free group as a subgroup. Further, every group is the quotient of some free group by a normal subgroup. A natural question arises whether such properties translate to the group C*-algebra of free groups.

Finite-dimensional C*-algebras are easily understood as direct sums of matrix algebras. In trying to understand arbitrary C*-algebras, it is therefore natural to approximate them, whenever possible, with finite-dimensional ones. This general strategy has led to the introduction of various important properties of C*-algebras, such as nuclearity and quasidiagonality. Perhaps the most basic finite-dimensional approximation property that a C*-algebra can enjoy is that of residual finite-dimensionality. The class of residually finite-dimensional (RFD) C*-algebras consists of those that can be embedded in a product of matrix algebras. In other words, RFD C*-algebras admit block-diagonal decompositions with finite-dimensional blocks ([8]).

Definition 2.1. (RFD C*-algebra) A C*-algebra is said to be residually finite dimensional if it admits a separating family of finite-dimensional representations.

We can look at this in another way. A representation π of a C*-algebra A over a Hilbert space H is said to be finite-dimensional if $\overline{\pi(A)H}$ is finite-dimensional. Note that in general, the finite-dimensionality of $\pi(A)$ is necessary but not sufficient for the finite-dimensionality of π. π is said to be RFD if it lies in the closure of all finite-dimensional representations in (A, H), the space of all (possibly degenerate) representations of A over H. The C*-algebra A is then RFD if every representation of it is RFD ([16]).

Clearly a C*-algebra is residually finite-dimensional if it is commutative or finite-dimensional. Also, this property is inherited by subalgebras. Menal and Goodearl proved in 1990 that any C*-algebra is a quotient of an RFD C*-algebra ([9]). In this section, we prove that the full C*-algebra of the free group on two generators also belongs to the class of RFD C*-algebras. We follow Choi's OG paper [10] that proved the result.

Theorem 2.2. *$C^*(\mathbb{F}_2)$ is residually finite dimensional.*

Proof. If π is the universal unitary representation of $\mathbb{F}_2$ on a Hilbert space H, then the full group C*-algebra $C^*(\mathbb{F}_2)$ is the C*-subalgebra of $B(H)$ generated by the set

$\{\pi(g) : g \in F_2\}$.
We may assume $C^*(\mathbb{F}_2) = C^*(U, V)$ where (U, V) is a universal pair of unitary operators (i.e. if you take any other pair of unitaries, the individual assignment gives you a $*$-homomorphism between the spanned C*-algebras).
Let $\{P_n\}$ be an increasing sequence of projections in $B(H)$ approaching to the identity operator in SOT, with rank of P_n being n.
Define $A_n = P_n U P_n$,
$B_n = P_n V P_n$,

$$U_n = \begin{bmatrix} A_n & (P_n - A_n A_n^*)^{1/2} \\ (P_n - A_n^* A_n)^{1/2} & -A_n^* \end{bmatrix},$$
$$V_n = \begin{bmatrix} B_n & (P_n - B_n B_n^*)^{1/2} \\ (P_n - B_n^* B_n)^{1/2} & -B_n^* \end{bmatrix}.$$

Note that U_n's and V_n's are unitaries due to Halmos' dilation (as stated in talk 5).
Now, identify $P_n H P_n$ with M_n.
So we may regards P_n as $I_{n\times n}$ and U_n and V_n as $2n \times 2n$ unitary matrices.
From the universal property of $C^*(\mathbb{F}_2)$, the assignment (U, V) to (U_n, V_n) defines a representation $\pi_n : C^*(\mathbb{F}_2) \to M_{2n}$.
TST (to show that) $\{\pi_n\}_{n=1}^{\infty}$ is a separating family of (finite dimensional) representations.
ETST (enough to show that) the $*$-homomorphism $\pi : C^*(\mathbb{F}_2) \to \prod_{n=1}^{\infty} M_{2n}$ defined by

$$\pi(S) = \prod_{n=1}^{\infty} \pi_n(S)$$

is actually $*$-injective.
We have the following convergences in SOT

$$U_n \to \begin{bmatrix} U & 0 \\ 0 & -U^* \end{bmatrix},\ U_n^* \to \begin{bmatrix} U^* & 0 \\ 0 & -U \end{bmatrix},\ V_n \to \begin{bmatrix} V & 0 \\ 0 & -V^* \end{bmatrix},\ V_n^* \to \begin{bmatrix} V^* & 0 \\ 0 & -V \end{bmatrix}$$

Hence, taking F to be the finite linear combination, we get the following convergence in SOT

$$F(U_n, V_n) \to \begin{bmatrix} F(U, V) & 0 \\ 0 & F(-U^*, -V^*) \end{bmatrix}.$$

Therefore, for any given $\epsilon > 0$ and given $||F(U, V)|| = 1$, we have $||F(U_n, V_n) \geq 1-\epsilon||$ for sufficiently large n.

$$\implies \|\pi(F(U, V))\| \geq \|\pi_n(F(U, V))\| = \|F(U_n, V_n)\| \geq 1 - \epsilon$$

Since ϵ is arbitrary, we conclude that π, restricted to the pre-C*-algebra (not necesssarily complete C*-algebra) generated by U, V is an isometry.
By continuity of π, it is an isometry and thus it is $*$-injective. $\square$

The following is a useful consequence of the above theorem.

Corollary 2.3. *$C^*(\mathbb{F}_2)$ admits a faithful tracial state.*

Proof. As seen in the proof of Theorem 2.2, we can imbed $C^*(\mathbb{F}_2)$ into $\prod_{n=1}^{\infty} M_{2n}$ as a C*-subalgebra.
Let τ_n be a faithful tracial state on M_{2n}.
Then we get the desired faithful tracial state from

$$\prod M_{2n} \longrightarrow \mathbb{C}$$
$$\prod S_n \longmapsto \sum \frac{\tau_n(S_n)}{2^n}$$

$\square$

3 RF $\overset{?}{\Longleftrightarrow}$ RFD

If a group is finitely residual,
Is its algebra residually finite dimensional?
The converse seems irrefutable
Is it hypothetical to assert they're congenial?
To the land of representation theory we travel
For the obstruction to unravel.

We restrict ourselves to finitely generated discrete groups in this section. This is because for such groups being residually finite is equivalent to having a separating family of finite dimensional unitary representations ([12]).

Theorem 3.1. *Let Γ be a finitely generated discrete group. If $C^*(\Gamma)$ is residually finite dimensional then Γ is residually finite.*

Proof. $C^*(\Gamma)$ is RFD implies that $C^*(\Gamma) \subset \prod_{i=1}^{\infty} M_{n_i}$
$\implies$ Γ embeds in $\prod_{i=1}^{\infty} U_{n_i}$ since $C^*(\Gamma)$ contains a copy of Γ.. $\square$

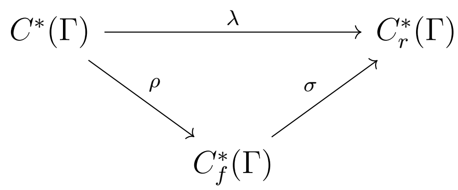
At first sight, it appears that the converse should be true. That is every residually finite group should give us a residually finite dimensional C*-algebra. In [13], Pierre de la Harpe provides a counter-example for this - $SL(2, \mathbb{Z}[1/p])$. In [15] and [14], Bekka provides classes of groups for which the converse does not hold. For example, groups which have a subgroup that does not have propety(T) and such that the trivial representation of the subgroup is isolated in the set of all finite dimensional unitary representations of the group do not give residually finite dimensional C*-algebras. de la Harpe further asks whether we can slightly alter the conditions to get an affirmative answer. Spronk and Wood answered this question in [11] by exhibiting the implication for the class of amenable groups.

Theorem 3.2. *Let Γ be an amenable group. Γ is residually finite iff $C^*(\Gamma)$ is residually finite dimensional.*

Proof. $(\implies)$
Denote by Σ the set of equivalence classes of finite-dimensional irreducible unitary representations of Γ.
Due to Malcev's theorem, Γ is residually finite iff Σ is a separating family of Γ.
Note that Γ is residually finite implies that the left regular representation λ of Γ is weakly contained in Σ, i.e.

$$\ker \lambda \supset \bigcap_{\pi_\Gamma \in \Sigma} \ker \pi$$

We refer the reader to [17] Example 1.11 (ii) for the proof of the above statement.
Now, let $\rho_\Gamma = \prod_{\pi_\Gamma \in \Sigma} \pi_\Gamma$ and $C_f^*(\Gamma) = \rho(C^*(\Gamma))$, an RFD C*-algebra.
Define $\sigma : C_f^*(\Gamma) \to C_r^*(\Gamma)$ s.t. $\sigma(\rho(a)) = \lambda(a)$ for some a in $C^*(\Gamma)$.
Since λ_Γ is weakly contained in ρ_Γ, σ is well defined and is an involutive representation of $C_f^*(\Gamma)$. Hence the following diagram commutes.



Since Γ is amenable, due to Theorem 1.4 we have $C_r^*(\Gamma) \cong C^*(\Gamma) \cong C_f^*(\Gamma)$. $\square$

4 Kirchberg's conjectures

For some RF groups proving/disproving RFD is extremely hard – as proved by Kirchberg ([3]) the famous Connes Embedding Problem is equivalent to the question of whether or not $\mathbb{F}_2 \times \mathbb{F}_2$ is RFD (while it is well known to be RF).

Lemma 4.1. *Any trace (not necessarily faithful) on the maximal tensor product of two C*-algebras factors through their minimal tensor product.*

Proof. It is enough to prove the claim for extremal traces. These are traces which do not lie on any open line segment joining two points of the tracial simplex.
A trace is extremal iff its GNS representation generates a finite factor (i.e. a vN algebra with trivial center and such that every isometry is a unitary). The proof of this follows similar arguments as that of a state is pure iff its GNS representation is irreducible.
Let τ be an extremal tensor on $A \otimes_{max} B$ with the corresponding GNS representation being π which generates a finite factor M.
Let $\phi : A \to M$ and $\psi : B \to M$ be the restrictions of π.
$\implies M_A = \phi(A)''$ and $M_B = \psi(B)''$ are commuting vN algebras of M, which have to be factors.
Since a trace on a finite factor is unique, we get

$$\tau_M(ab) = \tau_{M_A}(a)\tau_{M_B}(b)$$

for $a \in A$ and $b \in B$.
$\implies M = M_A \overline{\otimes} M_B$ the vN tensor product. □

Theorem 4.2. *The following conjectures are equivalent.*

i. $C^*(\mathbb{F}_\infty) \otimes_{min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{max} C^*(\mathbb{F}_\infty)$

ii. $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty)$ *is residually finite dimensional.*

iii. $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty)$ *has a faithful tracial state.*

iv. The C-algebra* $C^*(\mathbb{F}_\infty)$ *has the weak expectation property (WEP).*

v. LLP implies WEP.

Proof. Note that $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{max} C^*(\mathbb{F}_\infty)$ canonically.
(i) $\implies$ (ii)
The map from the minimal tensor product to the maximal tensor product factors through an RFD C*-algebra. If the former two are isomorphic, then each is an RFD C*-algebra. This is similar to the proof of Theorem 3.2.
(ii) $\implies$ (iii) similarly as in the proof of Corollary 2.3.
(iii) $\implies$ (i) follows from Lemma 4.1 and the trace being faithful.
(iv) $\iff$ (i) was proved in talk 5 (Theorem 2.56 (2)). □

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