$C^*(\mathbb{F}) \otimes_{\max} \mathcal{B}(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\min} \mathcal{B}(\mathcal{H})$

Step 1 prove title
Step 2 deduce that exact $\Rightarrow$ nuclearly embeddable (maybe!)

Note every $A \hookrightarrow \mathcal{B}(\mathcal{H})$ and $C(\mathbb{F}) \twoheadrightarrow A$ for some $\mathcal{H}$ and $\mathbb{F}$.

§1 Operator spaces

An operator space E is a closed subspace of a C^*-algebra ...

E inherits a norm from the ambient C^*-alg A
$M_n(E)$ " " " " $M_n(A)$

Let $\varphi : E \to F$ be a linear map between operator spaces.
Then $\varphi_n : M_n(E) \to M_n(F) : (x_{ij}) \mapsto \varphi(x_{ij})$ is also linear

Say φ is completely bounded (cb.) if

$$\|\varphi\|_{cb} = \sup_n \|\varphi_n\| < \infty$$

Say φ is completely contractive if $\|\varphi\|_{cb} \leq 1$
completely isometric if $\forall n$ φ_n is an isometric

1 Thm (Wittstock, Haagerup, Paulsen)

Let $E \subseteq \mathcal{B}(\mathcal{H})$ $F \subseteq \mathcal{B}(K)$ be operator spaces

Every cb. map factorises as

$$\varphi(x) = V_2 \pi(x) V_1$$

where $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\hat{\mathcal{H}})$ is a $*$-rep
and $V_1 : K \to \hat{\mathcal{H}}$ and $V_2 : \hat{\mathcal{H}} \to K$ are bounded linear maps
Moreover, $\|\varphi\|_{cb} = \|V_1\|\|V_2\|$

12 Cor
Let $E \subseteq \tilde{E} \subseteq \mathcal{B}(\mathcal{H})$ operator spaces
$\varphi : E \to \mathcal{B}(K)$ cb.
Then $\exists\, \tilde{\varphi} : \tilde{E} \to \mathcal{B}(K)$ with $\tilde{\varphi}|_E = \varphi$ and $\|\tilde{\varphi}\|_{cb} = \|\varphi\|$

Proof
$V_2 \pi(x) V_1$ is defined for all $x \in \mathcal{B}(\mathcal{H})$

□

3 Cor

If $E \subseteq \mathcal{B}(\mathcal{H})$ op space $1 \in E$
$\varphi : E \to \mathcal{B}(K)$ cb. with $\|\varphi\|_{cb} = 1$ and $\varphi(1) = 1$

then $\varphi(x) = S^* \pi(x) S$
for some isometric $S : \mathcal{H} \to \hat{\mathcal{H}}$ and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\hat{\mathcal{H}})$ $*$-rep.

Proof

$\varphi(x) = V_2 \pi(x) V_1$ $\|V_1\| \|V_2\| = 1$
Wlog both are 1.

$1 = V_2 V_1$

$\|V_1(x)\| = \|x\|$ so V_1 is isometric embedding $K \hookrightarrow \mathcal{H}$
V_2^* also an isometric embedding $K \hookrightarrow \mathcal{H}$
Replacing π with $Ad(u) \circ \pi$ wlog $V_2^* = V_1$.

□

Def

operator spaces $E \subseteq \mathcal{B}(\mathcal{H})$ $F \subseteq \mathcal{B}(K)$
$E \otimes_{min} F := \overline{E \odot F}^{\mathcal{B}(\mathcal{H} \otimes K)}$

So $E \otimes_{min} F \subseteq \mathcal{B}(\mathcal{H}) \otimes_{min} \mathcal{B}(K)$

Useful Inequality

$$\| \sum_{i=1}^n a_i b_i \| \le \| \sum_{i=1}^n a_i a_i^* \|^{1/2} \| \sum_{i=1}^n b_i^* b_i \|^{1/2}$$

Proof

Set $a = \begin{pmatrix} a_1 & \cdots & a_n \\ & 0 & \end{pmatrix}$ $b = \begin{pmatrix} b_1 & \\ \vdots & 0 \\ b_n & \end{pmatrix}$

Then
$\text{LHS} = \|ab\| \le \|a\| \|b\| \le \|aa^*\|^{1/2} \|b^* b\|^{1/2}$.

! technical :

4 Proposition

If $\mathcal{H}$ Hilbert space $\dim(\mathcal{H}) = \infty$
$\mathbb{F}$ free group generated by $(U_i)_{i \in I}$
$(x_i)_{i \in I}$ finitely support family in $\mathcal{B}(\mathcal{H})$,

then

$$\left\| \sum_i U_i \otimes x_i \right\|_{C^*(\mathbb{F}) \otimes_{min} \mathcal{B}(\mathcal{H})} = \inf \left\{ \left\| \sum_i y_i y_i^* \right\|^{1/2} \left\| \sum_i z_i^* z_i \right\|^{1/2} : \begin{array}{l} y_i, z_i \in \mathcal{B}(\mathcal{H}) \\ x_i = y_i z_i \\ \forall i \end{array} \right\}$$

Proof

"$\leq$" follows from the usefull inequality with
$a_i = U_i \otimes y_i \quad b_i = 1 \otimes z_i$

"$\geq$"

$$\text{LHS} = \sup \left\{ \left\| \sum_i u_i \otimes x_i \right\|_{\mathcal{B}(K) \otimes_{min} \mathcal{B}(\mathcal{H})} : \begin{array}{l} K \text{ Hilbert space} \\ u_i \in \mathcal{U}(K) \\ \forall i \end{array} \right\}.$$

by universality of $C^*(\mathbb{F})$

$$= \sup \left\{ \left\| \sum_i t_i \otimes x_i \right\|_{\mathcal{B}(K) \otimes_{min} \mathcal{B}(\mathcal{H})} : \begin{array}{l} K \text{ HS} \\ t_i \in \mathcal{B}(K) \ \|t_i\| \leq 1 \end{array} \right\}.$$

by the Russo-Dye Thm:
$\overline{\text{conv}}\, \mathcal{U}(K) = \mathcal{B}(\mathcal{H})_1$
Moreover we can restrict to $\dim(K) < \infty$.

$$= \|T\|_{cb}$$

where $T : \ell^\infty(I) \to \mathcal{B}(\mathcal{H})$
$(\alpha_i) \mapsto \sum_i \alpha_i x_i$

(Thm 1,1) Factorise $T\alpha = V^* \pi(\alpha) W$ where $\pi : \ell^\infty(I) \to \mathcal{B}(\hat{\mathcal{H}})$
Let $(e_i)_{i \in I}$ be the "basis" of $\ell^\infty(I)$ " a $*$-rep.

Set $y_i = V^* \pi(e_i)$ and $z_i = \pi(e_i) W$.

Then $y_i z_i = T(e_i) = x_i$

$$\sum_i y_i y_i^* = V^* \pi\left(\sum_i e_i\right) V \leq V^* \cdot 1 \cdot V$$

$$\sum_i z_i^* z_i \leq W^* W$$

So $\left\| \sum_i y_i y_i^* \right\|^{1/2} \left\| \sum_i z_i^* z_i \right\|^{1/2} \leq \|V\| \|W\| = \|T\|_{cb}$.

□

Remark: It's convient to at $1 \in \mathbb{F}$ to the family of generators $(u_i)_{i \in I}$. The Proposition is still true as

$$\| u_0 \otimes \alpha_0 + \textstyle\sum_i' u_i \otimes \alpha_i \| = \| 1 \otimes x_0 + \textstyle\sum_i' u_0^{-1} u_i \otimes x_i \|$$

Proposition

If $\mathcal{H}$ Hilbert space $\dim \mathcal{H} = \infty$

$\mathbb{F}$ free group generators $(u_i)_{i \in I}$ s.t. $u_0 = 1$

E_1 closed span of $(u_i)_{i \in I}$.

$$E_1 \otimes_{\min} \mathcal{B}(\mathcal{H}) \underset{\substack{\text{completely} \\ \text{isometric}}}{\hookrightarrow} C^*(\mathbb{F}) \otimes_{\max} \mathcal{B}(\mathcal{H})$$

Proof

Let $x = \sum_i' u_i \otimes x_i$ for some finitely support family (x_i).

Suppose $\|x\|_{\min} < 1$.

Then $\exists\, y_i, z_i \in \mathcal{B}(\mathcal{H})$ s.t. $x_i = y_i z_i$ and

$$\| \textstyle\sum y_i y_i^* \| < 1, \quad \| \textstyle\sum_i' z_i z_i^* \| < 1$$

Let $\pi : C^*(\mathbb{F}) \odot \mathcal{B}(\mathcal{H}) \to \mathcal{B}(K)$ be a rep.
Let π_1, π_2 be the restrictions to $C^*(\mathbb{F}) \odot 1$ and $1 \odot \mathcal{B}(\mathcal{H})$.

Then

$$\begin{aligned} \pi(x) &= \textstyle\sum_i' \pi_1(u_i) \pi_2(x_i) \\ &= \textstyle\sum_i' \pi_1(u_i) \pi_2(y_i) \pi_2(z_i) \\ &= \textstyle\sum_i' \pi_2(y_i) \pi_1(u_i) \pi_2(z_i) \end{aligned} \qquad \pi_1, \pi_2 \text{ commute.}$$

Useful inequality.

$$\begin{aligned} \|\pi(x)\| &\le \| \textstyle\sum_i' \pi_2(y_i) \pi_2(y_i)^* \| \, \| \textstyle\sum_i' \pi_2(z_i)^* \cancelto{1}{\pi_1(u_i)^* \pi_1(u_i)} \pi_2(z_i) \| \\ &= \| \pi_2(\textstyle\sum_i' y_i y_i^*) \| \, \| \pi_2(\textstyle\sum_i' z_i^* z_i) \| \\ &\le 1. \end{aligned}$$

So $\|x\|_{\max} \le 1$

Hence $E_1 \otimes_{\min} \mathcal{B}(\mathcal{H}) \hookrightarrow C^*(\mathbb{F}) \otimes \mathcal{B}(\mathcal{H})$ max.
matrix inflations get absorbed into $\mathcal{H}$.

6 <u>Lemma</u>

Let $u \in \mathcal{U}(\mathcal{H})$ $\hat{u} \in \mathcal{U}(\hat{\mathcal{H}})$
Suppose $u = S^*\hat{u}S$ for some isometry $S : \mathcal{H} \to \hat{\mathcal{H}}$

Then $\hat{u}$ commutes with the range projection SS^*.

<u>Proof</u>
View $\mathcal{H} \subseteq \hat{\mathcal{H}}$. Write $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}^\perp$.
Then

$$\hat{u} = \begin{pmatrix} u & b \\ c & d \end{pmatrix} \qquad S^*S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using $\hat{u}\hat{u}^* = 1$ we get $uu^* + bb^* = 0$ so $b = 0$.
Using $\hat{u}^*\hat{u} = 1$. $u^*u + c^*c = 0$ so $c = 0$.

$\square$

7 <u>Prop</u>
C^*-alg A, B
A generated by unitaries $(u_i)_{i \in I}$
E linear span of $(u_i)_{i \in I}$ and 1

Suppose $T : E \to B$ satisfies
$\|T\|_{cb} \leq 1$
$T(1) = 1$
$T(u_i) \in \mathcal{U}(B)$ $\forall i$
Then T extends to a $*$-hom $A \to B$.

<u>Proof</u>
Wlog $B = \mathcal{B}(\mathcal{H})$

Extend T to $\tilde{T} : A \to B$ with $\|\tilde{T}\|_{cb} = \|T\|$
Since $\|\tilde{T}\|_{cb} \leq 1$ and $\tilde{T}(1) = 1$

$$\tilde{T}a = S^* \tilde{\pi}(a) S$$

for some $*$-rep. $A \to \mathcal{B}(\hat{\mathcal{H}})$ and isometry $S : \mathcal{H} \to \hat{\mathcal{H}}$

Since $\tilde{T}(u_i) \in \mathcal{U}(B)$ $\forall i$

$\tilde{\pi}(u_i)$ commutes with S^*S $\forall i$ by Lemma 6.

$\therefore$ $\tilde{\pi}(A)$ commutes with S^*S

$\therefore$ $\tilde{T}(a)\tilde{T}(b) = T(ab)$.

$\square$

8 **Thm**
$$C^*(\mathbb{F}) \otimes_{min} \mathcal{B}(\mathcal{H}) \cong C^*(\mathbb{F}) \otimes_{max} \mathcal{B}(\mathcal{H})$$

Proof
Wlog $\dim \mathcal{H} = \infty$.

Let E be the operator space generated by (u_i) and 1.

By Prop 5.

$$\iota : E \otimes_{min} \mathcal{B}(\mathcal{H}) \hookrightarrow C^*(\mathbb{F}) \otimes_{max} \mathcal{B}(\mathcal{H})$$ completely isometric.

Since $\|\iota\|_{cb} \leq 1$ $\quad \iota(1) = 1$ $\quad \iota(u_i)$ is a unitary $\forall i$

ι extends to a $*$-hm $C^*(\mathbb{F}) \otimes_{min} \mathcal{B}(\mathcal{H}) \longrightarrow C^*(\mathbb{F}) \otimes_{max} \mathcal{B}(\mathcal{H})$
by prop 7.

So $\|x\|_{max} \leq \|x\|_{min} \quad \forall x \in C^*(\mathbb{F}) \odot \mathcal{B}(\mathcal{H})$

□

Exactness $\Rightarrow$ Nuclear embeddability

Recall

$\varphi : A \to B$ is nuclear iff $\forall C$ $\varphi \otimes_{max} id_C$ factors through $\varphi \otimes_{min} id_C$

Prop

Let $\varphi : A \to \mathcal{B}(\mathcal{H})$ be a $*$-rep.
Then
$\varphi \otimes_{max} id_{C^*(\mathbb{F})}$ factors through $\pi \otimes_{min} id_{C^*(\mathbb{F})}$

Proof

$$\begin{tikzcd}
A \otimes_{max} C^*(\mathbb{F}) \arrow[r, "\varphi \otimes_{max} id"] \arrow[d, "\Phi"'] & \mathcal{B}(\mathcal{H}) \otimes_{max} C^*(\mathbb{F}) \arrow[d, phantom, sloped, "\cong"] \\
A \otimes_{min} C^*(\mathbb{F}) \arrow[r, "\varphi \otimes_{min} id"] & \mathcal{B}(\mathcal{H}) \otimes_{min} C^*(\mathbb{F})
\end{tikzcd}$$

□

Thm

Let $\varphi : A \to \mathcal{B}(\mathcal{H})$ be a $*$-rep and A exact.
Then φ is nuclear

Proof

Let C be a C^* alg. Write $B = \mathcal{B}(\mathcal{H})$

Choose a exact sequence using universality of $C^*(\mathbb{F})$

$$0 \to J \to C^*(\mathbb{F}) \to C \to 0$$

Since A is exact,

$$0 \to A \otimes J \to A \otimes C^*(\mathbb{F}) \to A \otimes C \to 0$$

is exact.

We have.

$$\begin{tikzcd}
 & & A \otimes_{max} C \arrow[rr, "\varphi \otimes id_C"] & & B \otimes_{max} C \\
 & A \otimes_{max} C^*(\mathbb{F}) \arrow[ur, "id_A \otimes \pi"] \arrow[rr, "\varphi \otimes_{max} id"] \arrow[d, two heads] & & B \otimes_{max} C^*(\mathbb{F}) \arrow[ur, "id_B \otimes \pi"'] & \\
A \otimes_{min} J \arrow[r] & A \otimes_{min} C^*(\mathbb{F}) \arrow[urr, dashed, "\Psi"] \arrow[r, "id \otimes \pi"] & A \otimes_{min} C & &
\end{tikzcd}$$

Ψ exists by prev prop

$\Phi = id_B \circ \pi \circ \Psi$ vanishes on $A \odot J$

So on $A \otimes_{min} J$ there is an induce map

$\overline{\Phi} : A \otimes_{min} C \longrightarrow B \otimes_{max} C.$

So $\varphi \otimes_{max} id_C$ factor through $\varphi \otimes_{min} id_C$.

∴ φ is nuclear

□.