The Local Lifiting Property

Throughout all C^*-algebras are assumed to be unital. (In the nonunital setting, we pass to the minimal unitization.)

Definition 2.36. *Let A and B be C^*-algebras and J a closed two-sided ideal in B with quotient map $\pi : B \to B/J$. A ucp map $\phi : A \to B/J$ is* ***liftable*** *if there is a ucp map $\psi : A \to B$ such that $\pi \circ \psi = \phi$. If A is unital, we say a ucp map $\phi : A \to B/J$ is* ***locally liftable*** *if for any finite dimensional operator system $S \subseteq A$ there is a ucp map $\psi : S \to B$ such that $\pi \circ \psi = \phi|_S$.*

Theorem 2.37 (Choi-Effros)**.** *Let A and B be C^*-algebras and J a closed two-sided ideal in B with quotient map $\pi : B \to B/J$. Any nuclear map[9] $\rho : A \to B/J$ is liftable.*

Remark 2.38. *Separable is required here. Since $c_0(\mathbb{R})$ (treating $\mathbb{R}$ as an uncountable index set) embeds into $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$, if this embedding lifted, it would lift to an embedding of $c_0(\mathbb{R})$ into $\ell^\infty(\mathbb{N})$ as a Banach space, which can't happen since $\ell^\infty(\mathbb{N})$ has a separable predual. This map does locally lift, but we'll get back to that.*

Definition 2.39 (LLP)**.** *A unital C^* algebra A has the* ***(local) lifting property*** *(L)LP if any ucp map from A into a quotient C^*-algebra is (locally) liftable.*

Example 2.40. *Any nuclear C^*-algebra has the LLP. (We'll return to this.) Any separable nuclear C^*-algebra has the LP.*

Theorem 2.41 (Kirchberg)**.** *For any free group $\mathbb{F}$, $C^*(\mathbb{F})$ has the LLP. If $\mathbb{F}$ is countably generated, then $C^*(\mathbb{F})$ has the LP.*

Lemma 2.42. *Let $\pi : A \to B$ be a $*$-homomorphism between C^*-algebras and $b \in \pi(A)$. Show that there exists $a \in A$ with $\pi(a) = b$ and $\|a\| = \|b\|$.*

This works in the non-unital setting too via unitizations.

Proof. It suffices to consider $\|b\| = 1$. Let $a \in A$ with $\pi(a) = b$. Then $1 = \|\pi(a)\| \leq \|a\|$. Assume $\|a\| > 1$. Then $\||a|\| > 1$. By taking a faithful representation, we assume $A \subset B(\mathcal{H})$ and let $a = u|a|$ be the polar decomposition of a in $B(\mathcal{H})$. Define $f \in C[0, \|a\|]$ by

$$f(t) = \begin{cases} t & ; \quad t \in [0,1] \\ 1 & ; \quad t \in (1, \|a\|] \end{cases}.$$

Note that $uf(|a|) \in A$.[10] Moreover,

$$\|uf(|a|)\| \leq \|u\|\|f(|a|)\| = \|f(|a|)\| \leq 1.$$

We claim $uf(|a|)$ is the desired lift of b. To that end, define $g \in C([0, \|a\|])$ by

$$g(t) = \begin{cases} 1 & ; \quad t \in [0,1] \\ t^{-1} & ; \quad t \in (1, \|a\|] \end{cases}.$$

Then $uf(|a|) = u|a|g(|a|) = ag(|a|)$, and so

$$\pi(uf(|a|)) = \pi(uf(|a|)) = \pi(ag(|a|)) = \pi(a)g(|\pi(a)|) = bg(|b|).$$

But as a continuous function in $C(\sigma(|b|))$, $g \equiv 1$. Hence $g(|b|) = 1$, and so $bg(|b|) = b$. So, $\pi(uf(|a|)) = b$ and moreover,

$$\|b\| \geq \|uf(|a|)\| \geq \|\pi(uf(|a|))\| = \|b\|.$$

□

[9] Recall that a map $\rho : A \to B$ between C^*-algebras is nuclear if there exists a net of matrix algebras M_{n_λ} and cpc maps $A \xrightarrow{\psi_\lambda} M_{n_\lambda} \xrightarrow{\varphi_\lambda} B$ so that $\varphi_\lambda \circ \psi_\lambda$ converges pointwise in norm to ρ.

[10] Indeed, by by Stone-Weierstraß, it suffices to prove that $up(|a|) \in A$ for any polynomial vanishing at 0. Let $p(z) = \sum_{k=1}^n \lambda_k z^k$. Then since $|a| \in A$, we have $up(|a|) = \sum_{k=1}^n \lambda_k u|a|^k = \sum_{k=1}^n \lambda_k a|a|^{k-1} \in A$.

Proposition 2.43 (Halmos Dilation)**.** *Let $a \in A$ with $\|a\| \leq 1$. Then*

$$\begin{pmatrix} a & (1-aa^*)^{1/2} \\ (1-a^*a)^{1/2} & -a \end{pmatrix} \in M_2(A)$$

is a unitary.

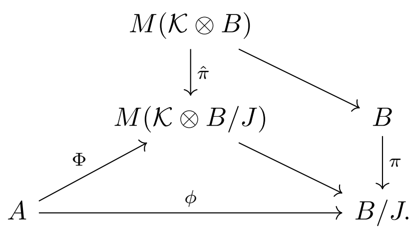
Theorem 2.44 (Noncommutative Tietze Extension Theorem)**.** *Let A and B be* C*-*algebras and $\pi : B \to A$ a surjective *-homomorphism. Then π extends uniquely to a *-homomorphism $\hat{\pi} : M(B) \to M(A)$ where $\hat{\pi}(x) \cdot \pi(b) = \pi(x \cdot b)$ for all $b \in B$. Moreover, if B is σ-unital, then π is surjective.*

Theorem 2.45 (Kasparov Stinespring Dilation)**.** *Let A and B be unital* C*-*algebras with A separable, and let $\phi : A \to B$ be a ucp map. Then there exists a unital *-homomophism $\Phi : A \to M(\mathcal{K} \otimes B)$ such that for each $a \in A$,*

$$(e_{11} \otimes 1_B)\Phi(a)(e_{11} \otimes 1_B) = e_{11} \otimes \phi(a).$$

Lemma 2.46. *Let $\mathbb{F}$ be a free group. For any finite dimensional subspace $D \subseteq C^*(\mathbb{F})$, there is a copy of* $\mathrm{C}^*(\mathbb{F}_\infty) \subset \mathrm{C}^*(\mathbb{F})$ *with* $D \subset \mathrm{C}^*(\mathbb{F}_\infty)$ *such that there is a conditional expectation* $\mathrm{C}^*(\mathbb{F}) \to \mathrm{C}^*(\mathbb{F}_\infty)$.

Proof of Theorem 2.41. Using Lemma 2.46, we reduce to the case where $\mathbb{F}$ has a countable generating set, i.e., $\mathrm{C}^*(\mathbb{F})$ is separable. Let B be a unital C^*-algebra and $J \lhd B$, and let $\phi : \mathrm{C}^*(\mathbb{F}) \to B/J$ be ucp. Since B is unital, $\mathcal{K} \otimes B/J$ is σ-unital. Identifying $e_{11} \otimes B \subset M(\mathcal{K} \otimes B)$ with B and $e_{11} \otimes B/J \subset M(\mathcal{K} \otimes B/J)$ with B/J, we can combine Theorems 2.44 and 2.45 to get a commutative diagram

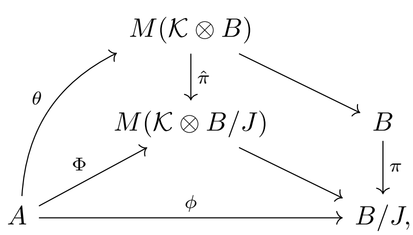


We have now effectively reduced the problem to asking whether any *-homomorphism from $\mathrm{C}^*(\mathbb{F})$ into a quotient lifts to a ucp map, and this is essentially what we show in the rest of the proof.

Writing $\{u_\lambda\}_\lambda$ for the generators of $\mathbb{F}$, each $\Phi(u_\lambda) \in M(\mathcal{K} \otimes B/J)$ is a contraction, and hence by Lemma 2.42 these lift to contractions $a_\lambda \in M(\mathcal{K} \otimes B)$, which we dilate to unitaries

$$v_\lambda := \begin{pmatrix} a_\lambda & (1-a_\lambda a_\lambda^*)^{1/2} \\ (1-a_\lambda^* a_\lambda)^{1/2} & -a_\lambda \end{pmatrix} \in M_2(M(\mathcal{K} \otimes B))$$

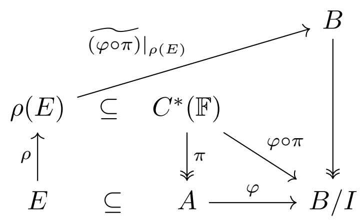
as in Proposition 2.43. The universal property of $\mathrm{C}^*(\mathbb{F})$ tells us the assignment $u_\lambda \mapsto v_\lambda$ induces a *-homomorphism $\psi : \mathrm{C}^*(\mathbb{F}) \to \mathrm{C}^*(\{v_\lambda\}_\lambda)$. Composing this with the compression $M_2(M(\mathcal{K} \otimes B)) \to M(\mathcal{K} \otimes B)$ denote the compression onto the 1,1 corner (a ucp map), we get a ucp map $\theta : A \to M(\mathcal{K} \otimes B)$ so that the following diagram commutes



yielding a ucp lift of ϕ. □

Proposition 2.47. *Let A be a C^*-algebra and $\mathbb{F}$ a free group such that A can be identified with a quotient $C^*(\mathbb{F})/J$ of $C^*(\mathbb{F})$. Then A has the LLP iff the identity on $C^*(\mathbb{F})/J$ is locally liftable. (If A is separable we can likewise characterize the LP.)*

Proof. For simplicity, identify $A = C^*(\mathbb{F})/J$, and let $\pi : C^*(\mathbb{F}) \to C^*(\mathbb{F})/J$ be the quotient map. Let $E \subseteq A$ be a finite dimensional operator system and $\rho : E \to C^*(\mathbb{F})$ the lift of $\mathrm{id}_A|_E$ guaranteed by assumption. Let B a C^*-algebra with closed two-sided ideal I, and $\varphi : A \to B/I$ a ucp map. Then, $\varphi \circ \pi : C^*(\mathbb{F}) \to B/I$ is a ucp map and $\rho(E) \subseteq C^*(\mathbb{F})$ is a finite dimensional operator system. Then, since $C^*(\mathbb{F})$ has the LLP, there is a lift $\widetilde{(\varphi \circ \pi)}|_{\rho(E)}$ of $(\varphi \circ \pi)|_{\rho(E)}$ to B.



□

Example 2.48. *Projective C^*-algebras have the LLP. (Separable projective C^*-algebras have the LP.)*

Outside the nuclear setting and easy corollaries to the above, examples and non-examples of the LLP are hard to come by. In [8], Ozawa proved the existence of groups whose full C*-algebra does not have the LLP, but no concrete examples were known until Andreas Thom's example of hyperlinear groups that are not residually finite ([10]).

Remark 2.49. *Because Thom does not spell out the argument for why these group* C*-*algebras lack the LLP, we sketch it here and outsource relevant definitions. For a group G, let τ_λ denote the trace on $C^*(G)$ coming from its left regular representation. We say a group G is hyperlinear if τ_λ is hyperlinear, meaning there exists a $*$-homomorphism $\rho : C^*(G) \to R^\omega$ such that $\tau_\lambda = \tau_\omega \rho$. We say G has Kirchberg's factorization property (F) if τ_λ is amenable (see* [1, Theorem 6.4.3]*), meaning there exists a $*$-homomorphism $\rho : C^*(G) \to R^\omega$ such that $\tau_\lambda = \tau_\omega \rho$ and such that ρ has a ucp lift to $\ell^\infty(R)$. The full* C*-*algebra of a hyperlinear group without (F) would fail to have the LLP. (Indeed, since R^ω is QWEP, if $C^*(G)$ had the LLP, then by* [7, Corollary 3.12]*, any ucp map into R^ω would have a ucp lift– not just local lift.) Kirchberg showed in* [5] *that in the presence of Kazhdan's Property (T), the factorization property (F) and residual finiteness (RF) are the same. Hence Thom's example is hyperlinear without property (F).*

Theorem 2.50 (Effros-Haagerup)**.** *Let B be a unital* C*-*algebra and $J \triangleleft B$. The following are equivalent*

(1) For any C*-*algebra C, the sequence*

$$0 \to C \otimes_{min} J \to C \otimes_{min} B \to C \otimes_{min} B/J \to 0$$

is exact.

(2) The sequence

$$0 \to B(\mathcal{H}) \otimes_{min} J \to B(\mathcal{H}) \otimes_{min} B \to B(\mathcal{H}) \otimes_{min} B/J \to 0$$

is exact for some infinite dimensional Hilbert space $\mathcal{H}$.

(3) $\mathrm{id}_{B/J}$ *is locally liftable.*

Corollary 2.51. *Let A be a unital C^*-algebra. Then, TFAE*

(1) A has the LLP.
(2) $A \otimes_{min} B(\mathcal{H}) = A \otimes_{max} B(\mathcal{H})$ for some infinite dimensional Hilbert space H.
(3) $A \otimes_{min} B(\mathcal{H}) = A \otimes_{max} B(\mathcal{H})$ for any infinite-dimensional Hilbert space H.

Proof. We start with some general observations. Fix a free group $\mathbb{F}$ and ideal $J \triangleleft \mathrm{C}^*(\mathbb{F})$ such that A can be identified with the quotient $\mathrm{C}^*(\mathbb{F})/J$.

Now, for any infinite-dimensional Hilbert space $\mathcal{H}$, since $\otimes_{\max}$ is exact and $C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H})$, it follows that $B(\mathcal{H}) \otimes_{\max} A \cong \frac{B(\mathcal{H})\otimes_{\min} C^*(\mathbb{F})}{B(\mathcal{H})\otimes_{\min} C^*(\mathbb{F})}$, as we can see from the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B(\mathcal{H}) \otimes_{\min} J & \hookrightarrow & B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}) & \twoheadrightarrow & \frac{B(\mathcal{H})\otimes_{\min} C^*(\mathbb{F})}{B(\mathcal{H})\otimes_{\min} C^*(\mathbb{F})} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & B(\mathcal{H}) \otimes_{\max} J & \hookrightarrow & B(\mathcal{H}) \otimes_{\max} C^*(\mathbb{F}) & \twoheadrightarrow & B(\mathcal{H}) \otimes_{\max} A & \longrightarrow & 0 \end{array}$$

Recall that the sequence

$$0 \to B(\mathcal{H}) \otimes_{\min} J \to B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}) \to B(\mathcal{H}) \otimes_{\min} A \to 0 \qquad (*)$$

is exact iff $\frac{B(\mathcal{H})\otimes_{\min} C^*(\mathbb{F})}{B(\mathcal{H})\otimes_{\min} C^*(\mathbb{F})} \cong B(\mathcal{H}) \otimes_{\min} A$. It follows that (*) is exact iff $B(\mathcal{H}) \otimes_{\min} A = B(\mathcal{H}) \otimes_{\max} A$.

On the other hand, the Effros-Haagerup lifting theorem tells us that the following are equivalent:

(1) The sequence

$$0 \to B(\mathcal{H}) \otimes_{\min} J \to B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}) \to B(\mathcal{H}) \otimes_{\min} A \to 0$$

is exact for any infinite dimensional Hilbert space $\mathcal{H}$.

(2) The sequence

$$0 \to B(\mathcal{H}) \otimes_{\min} J \to B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}) \to B(\mathcal{H}) \otimes_{\min} A \to 0$$

is exact for some infinite dimensional Hilbert space $\mathcal{H}$.

(3) id_A is locally liftable.

Combining this with the above observations, the following are equivalent:

(1) $B(\mathcal{H}) \otimes_{\min} A = B(\mathcal{H}) \otimes_{\max} A$ for any infinite dimensional Hilbert space $\mathcal{H}$.
(2) $B(\mathcal{H}) \otimes_{\min} A = B(\mathcal{H}) \otimes_{\max} A$ for some infinite dimensional Hilbert space $\mathcal{H}$.
(3) id_A is locally liftable.

Recalling from Proposition 2.47 that $\mathrm{id}_A : A \to C^*(\mathbb{F})/J$ locally lifts if and only if A has the LLP, we are done. □

Theorem 2.52 (Junge-Pisier)**.** *For any infinite-dimensional Hilbert space $\mathcal{H}$,*

$$B(H) \otimes_{max} B(H) \neq B(H) \otimes_{min} B(H).$$

It follows that $B(\mathcal{H})$ has the WEP but not the LLP. This actually disproved a number of conjectures that Kirchberg proved are equivalent in [3]:

Theorem 2.53 (Kirchberg)**.** *The following conjectures are equivalent:*

(1) $Ext(A)$ is a group for every separable unital C^*-algebra A with the WEP.
(2) Every finite dimensional operator system is unitally completely isometrically isomorphic to an operator system in $C^*(\mathbb{F}_\infty)$.
(3) For every pair A and B of separable unital C^*-algebras with the WEP,

$$A \otimes_{max} B = A \otimes_{min} B.$$

(4) The WEP and approximate injectivity are equivalent.

We have a similar tensor characterization for WEP. Recall that a C^*-algebra A has the WEP iff for any C^*-algebra B with $A \subseteq B$ and any C^*-algebra C,

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C.$$

Remark 2.54. *The proof of our earlier characterization of WEP actually shows something more general: for any pair $A \subset B$ there exists a weak conditional expectation $B \to A^{**}$ iff $A \otimes_{max} C \subset B \otimes_{max} C$ for any C. This is called a* relatively weakly injective embedding*, and a prime example is when there's an honest conditional expectation $B \to A$.*

An example comes from subgroups of discrete groups. For discrete group G with subgroup H, the inclusion $H \subset G$ induces a natural inclusion $C^*(H) \subset C^*(G)$, and there exists a conditional expectation $C^*(G) \to C^*(H)$. (See [9, Proposition 8.8]*.*

Corollary 2.55 (Kirchberg)**.** *For any C^*-algebra A, the following are equivalent*

(1) A has the WEP.
(2) For any embedding $A \subseteq B(H)$ and any free group $\mathbb{F}$

$$A \otimes_{max} C^*(\mathbb{F}) \subseteq B(H) \otimes_{max} C^*(\mathbb{F})$$

(3) For any embedding $A \subseteq B(H)$ and some non-abelian free group $\mathbb{F}$

$$A \otimes_{max} C^*(\mathbb{F}) \subseteq B(H) \otimes_{max} C^*(\mathbb{F}).$$

(4) $\mathrm{C}^*(\mathbb{F}) \otimes_{max} A = \mathrm{C}^*(\mathbb{F}) \otimes_{min} A$ *for any free group.*
(5) $\mathrm{C}^*(\mathbb{F}) \otimes_{max} A = \mathrm{C}^*(\mathbb{F}) \otimes_{min} A$ *for some non-abelian free group.*

Proof. Fix a faithful embedding $A \subseteq B(H)$.

For any free group $\mathbb{F}$, we have

$$A \otimes_{\min} \mathrm{C}^*(\mathbb{F}) \subset B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}) = B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}).$$

It follows that $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}) \subset B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F})$ iff $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}) = A \otimes_{\min} \mathrm{C}^*(\mathbb{F})$:

$$\begin{array}{ccc} B(H) \otimes_{\max} C^*(\mathbb{F}_\infty) & = & B(H) \otimes_{\min} C^*(\mathbb{F}_\infty) \\ \cup\ (\Leftrightarrow) & & \cup \\ A \otimes_{\max} C^*(\mathbb{F}_\infty) & \overset{(\Leftrightarrow)}{=} & A \otimes_{\min} C^*(\mathbb{F}_\infty) \end{array}$$

With that observation, and the previous characterization of the WEP, we have the following implications: (1) $\Rightarrow$ (2) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (3).

Next we show (3) $\Rightarrow$ (2). First we claim we may assume the "some" free group in (3) is $\mathbb{F}_\infty$. In particular, if $A \otimes_{\max} C^*(\mathbb{F}) \subseteq B(H) \otimes_{\max} \mathrm{C}^*(\mathbb{F})$ for some free group $\mathbb{F}$, then $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}) = A \otimes_{\min} \mathrm{C}^*(\mathbb{F})$, which implies $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) = A \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)$, and hence $A \otimes_{\max} C^*(\mathbb{F}_\infty) \subseteq B(H) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$. Indeed, since $\mathbb{F}_2$ contains a subgroup isomorphic to $\mathbb{F}_\infty$, all non-abelian free groups do. Then by Remark 2.54, this induces an embedding $\mathrm{C}^*(\mathbb{F}_\infty) \subset \mathrm{C}^*(\mathbb{F})$ and a conditional expectation $\mathrm{C}^*(\mathbb{F}) \to \mathrm{C}^*(\mathbb{F}_\infty)$. and it follows that $C \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \subset C \otimes_{\max} \mathrm{C}^*(\mathbb{F})$ for any C*-algebra C. In particular, if $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}) = A \otimes_{\min} \mathrm{C}^*(\mathbb{F})$, then $A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) = A \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)$.

So, we assume that the "some" free group for which (3) holds is $\mathbb{F}_\infty$. Now, let $\mathbb{F}$ be any other free group and $t = \sum_{i=1}^n a_i \otimes x_i \in A \odot C^*(\mathbb{F})$. Now with 2.46, with Remark 2.54, it follows that for any C*-algebra C (in particular, $C = A$ and $C = B(\mathcal{H})$)

$$C \otimes_{\max} C^*(\mathbb{F}_\infty) \subseteq C \otimes_{\max} C^*(\mathbb{F}).$$

Interpreting this in terms of norms, we have

$$\|t\|_{A \otimes_{\max} C^*(\mathbb{F})} = \|t\|_{A \otimes_{\max} C^*(\mathbb{F}_\infty)} = \|t\|_{B(H) \otimes_{\max} C^*(\mathbb{F}_\infty)} = \|t\|_{B(H) \otimes_{\max} C^*(\mathbb{F})}.$$

It remains to show (2) $\Rightarrow$ (1). Embed $\pi_u(A) \subseteq A^{**} \subseteq B(H_u)$ where $(\pi_u, B(H_u))$ is the universal representation for A. Let $\mathbb{F}$ be a free group on $|\mathcal{U}(\pi_u(A)')|$-many generators, and let $\pi : C^*(\mathbb{F}) \to \pi_u(A)' \subseteq B(H_u)$ be the surjective *-homomorphism induced by mapping generators of $\mathbb{F}$ to the unitaries of $\pi_u(A)'$. Now, we use The Trick[11] with $C = C^*(\mathbb{F})$, $B = B(H_u)$, $B(H) = B(H_u)$, $\pi_A = \pi_u$, and $\pi_C = \pi$. Then, we get a cpc map $\phi : B(H_u) \to \pi_u(A)'' = A^{**}$. □

Theorem 2.56 (Kirchberg,[3])**.** *Let A and B be C^*-algebras. Then*

(1) A has the LLP $\Leftrightarrow$ $A \otimes_{max} B(\ell^2) = A \otimes_{min} B(\ell^2)$,
(2) B has the WEP $\Leftrightarrow$ $C^*(\mathbb{F}_\infty) \otimes_{max} B = C^*(\mathbb{F}_\infty) \otimes_{min} B$, and
(3) A has the LLP and B has the WEP $\Rightarrow$ $A \otimes_{max} B = A \otimes_{min} B$.

Proof. It remains to show (3). To that end, faithfully embed $B \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since B has the WEP, $A \otimes_{\max} B \subseteq A \otimes_{\max} B(\mathcal{H})$; since A has the LLP, $A \otimes_{\max} B(\mathcal{H}) = A \otimes_{\min} B(\mathcal{H})$. Therefore, we have

$$A \otimes_{\max} B \subseteq A \otimes_{\max} B(\mathcal{H}) = A \otimes_{\min} B(\mathcal{H}) \supseteq A \otimes_{\min} B.$$

In other words, the topologies (and hence norms) agree. □

[11]¡The Trick: Let $A \subseteq B$ and C be C^*-algebras such that $A \otimes_{max} C \subseteq B \otimes_{max} C$. Given representations $\pi_A : A \to B(H)$ and $\pi_C : C \to B(H)$ with commuting ranges, there exists a ccp map $\phi : B \to \pi_C(C)'$ which extends π_A.

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